

# THE PERMANENT FUNCTION AS AN INNER PRODUCT<sup>1</sup>

BY MARVIN MARCUS AND MORRIS NEWMAN

Communicated by A. M. Gleason, November 25, 1960

Let  $U$  be a unitary  $n$ -space with inner product  $(x, y)$  and let  $U^{(k)}$  denote the space of totally symmetric multilinear functionals of  $k$ -tuples of vectors from  $U$ . If  $x_1, \dots, x_k$  are vectors in  $U$ , then the element  $\phi = x_1 \otimes \dots \otimes x_k \in U^{(k)}$  is defined by

$$\phi(u_1, \dots, u_k) = \sum_{\sigma} \prod_{i=1}^k (u_i, x_{\sigma(i)})$$

where the summation extends over all  $k!$  permutations  $\sigma$  of  $1, \dots, k$ . The conjugate bilinear functional

$$(1) \quad (x_1 \otimes \dots \otimes x_k, y_1 \otimes \dots \otimes y_k) = \text{per}[(x_i, y_j)] \quad i, j = 1, \dots, k$$

is an inner product in  $U^{(k)}$ . Here  $\text{per}(A)$  denotes the permanent of  $A$ ,

$$\text{per}(A) = \sum_{\sigma} \prod_{i=1}^k a_{i\sigma(i)}.$$

By applying the Schwarz inequality to the inner product (1) and making a careful investigation of the case of equality we obtain the

**THEOREM.** *If  $x_1, \dots, x_k$  and  $y_1, \dots, y_k$  are two sets of  $k$  vectors each, then*

$$(2) \quad |\text{per}[(x_i, y_j)]|^2 \leq \text{per}[(x_i, x_j)] \text{per}[(y_i, y_j)].$$

*If the zero vector occurs in neither set then equality in (2) implies that the space spanned by both sets is the same. If the sets are linearly independent then equality in (2) implies that except for order and scalar factors the sets are termwise identical.*

A number of corollaries follow easily. The first of these is a partial answer to a conjecture of van der Waerden [1].

I. *If  $A$  is an  $n$ -square doubly stochastic positive semi-definite symmetric matrix then*

$$(3) \quad \text{per}(A) \geq n!/n^n$$

*with equality if and only if  $A$  is the matrix all of whose entries are  $1/n$ .*

II. *If  $A$  and  $B$  are  $n$ -square complex matrices then*

---

<sup>1</sup> This work was supported in part by the Office of Naval Research.