ON DIFFERENTIABLE FUNCTIONS

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1. Let $f(x)$ be a real function defined over the closed interval $[a, b]$ and differentiable at each point of the interval. The following is a well-known classical theorem.

DARBOUX'S THEOREM. *If $\xi, \eta \in [a, b]$ and $f'(\xi) = c$, $f'(\eta) = d$ and $c < d$, then, if $c < k < d$, there exists at least one point ζ lying between ξ and η such that $f'(\zeta) = k$.*

Recently in 1947, Clarkson [1] developed the above property into the following important theorem concerning the behavior of the derivative $f'(x)$.

CLARKSON'S THEOREM. *Let α, β with $\alpha < \beta$ be any two real numbers and let us denote the aggregate of points in $[a, b]$ such that $\alpha < f'(x) < \beta$ by*

$$E_{\alpha\beta} = E(x; \alpha < f'(x) < \beta);$$

then $E_{\alpha\beta}$ is either void or $\mathfrak{M}(E_{\alpha\beta}) > 0$, where $\mathfrak{M}(E)$ is the Lebesgue measure of the set E.

2. In this note, we intend to give a more detailed description of $E_{\alpha\beta}$. We prove the following

THEOREM. *The set $E_{\alpha\beta}$ gives rise to a set of nonoverlapping and nonabutting open sub-intervals $\{I_i\}$ in the space $[a, b]$ and a closed set G, which is the complementary set of $\{I_i\}$ with respect to $[a, b]$, such that $E_{\alpha\beta}$ is void in each I_i and metrically dense everywhere in G.*

3. We first show that $E_{\alpha\beta}$ is metrically dense in itself. For, take any point P of $E_{\alpha\beta}$ and any neighborhood U_P containing P as its interior point, since $U_P \cap E_{\alpha\beta}$ is not void, from Clarkson's theorem, $\mathfrak{M}(U_P \cap E_{\alpha\beta}) > 0$. Thus, each point of $E_{\alpha\beta}$ is a limiting point of the set. I.e., $E_{\alpha\beta} \subseteq E'_{\alpha\beta}$. Moreover, for each point $P \in CE'_{\alpha\beta}$, it is always possible for us to construct a largest open sub-interval I_P relative to the space $[a, b]$ such that it contains P as its interior point and has its two end points belonging to the set $E'_{\alpha\beta}$,[1] and such that all of its interior points are the points of $CE'_{\alpha\beta}$. Thus, $E_{\alpha\beta}$ is void in each I_P, since $E_{\alpha\beta} \subseteq E'_{\alpha\beta}$. Hence, we see that, corresponding to the set

[1] In case $a \in CE'_{\alpha\beta}$ (similarly for b), the corresponding U_a takes a as one of its end points and the other end point of U_a is of course a point of $E'_{\alpha\beta}$.