AN EXACT SEQUENCE IN DIFFERENTIAL TOPOLOGY

MORRIS W. HIRSCH

Communicated by Deane Montgomery, May 12, 1960

1. **Introduction.** The purpose of this note is to describe an exact sequence relating three series of abelian groups: Γ^n, defined by Thom [3]; θ^n, defined by Milnor [1]; and Λ^n, defined below. The sequence is written

$$(1) \qquad \cdots \longrightarrow \Gamma^n \xrightarrow{j} \theta^n \xrightarrow{k} \Lambda^n \xrightarrow{d} \Gamma^{n-1} \longrightarrow \cdots.$$

We now describe these groups briefly.

To obtain Γ^n, divide the group of diffeomorphisms of the $n-1$ sphere S^{n-1} by the normal subgroup of those diffeomorphisms that are extendable to the n-ball. See [2] for details.

The set θ^n is the set of J-equivalence classes of closed, oriented, differentiable n-manifolds that are homotopy spheres. If M is an oriented manifold, let $-M$ be the oppositely oriented manifold. Two closed oriented n-manifolds M and N are *J-equivalent* if there is an oriented $n+1$-manifold X whose boundary is the disjoint union of M and $-N$, and which admits both M and N as deformation retracts. We denote the J-equivalence class of M by $[M]$. If $[M]$ and $[N]$ are elements of θ^n, their sum is defined to be $[M \# N]$, where $[M \# N]$ is obtained by removing the interior of an n-ball from M and N and identifying the boundaries in a suitable way. Details may be found in [1].

The group Λ^n is defined analogously using combinatorial manifolds. Instead of the interior of an n-ball, the interior of an n-simplex is removed. If M is a combinatorial manifold, we write $\langle M\rangle$ for its J-equivalence class.

2. **The sequence.** To define $k\colon \theta^n \to \Lambda^n$, we observe that every differentiable manifold M defines a combinatorial manifold $\overline{M}$, unique up to combinatorial equivalence, by means of a smooth triangulation of M [4]. We define $k[M]=\langle\overline{M}\rangle$.

Let $g\colon S^{n-1}\to S^{n-1}$ represent an element γ of Γ^n. According to J. Munkres [2], there is a unique (up to diffeomorphism) differentiable manifold V_γ corresponding to γ, such that $\overline{V}_\gamma=\overline{S}^n$. To obtain V_γ, identify two copies of R^n-0 by the diffeomorphism $x\to(1/|x|)g(x/|x|)$. Here R^n is Euclidean n-space and $|x|$ is the usual norm. The diffeomorphism class of V_γ depends only on γ, and