ASYMPTOTIC DISTRIBUTION OF EIGENVALUES OF BLOCK TOEPLITZ MATRICES

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Let $g(\lambda)$, $-\pi \leqq \lambda \leqq \pi$, be a $p \times p$ ($p=1, 2, \cdots$) matrix-valued Hermitian function. Further $g(\lambda)$ is bounded on $[-\pi, \pi]$, that is, its elements are bounded on $[-\pi, \pi]$. The Fourier coefficients

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\lambda} g(\lambda) d\lambda, \qquad k = 0, \pm 1, \cdots, \tag{1}$$

are then bounded in k. We call the $np \times np$ matrix

$$A_n = (a_{j-k}; j, k = 1, \cdots, n)$$

(an $n \times n$ matrix of the $p \times p$ blocks a_{j-k}) the nth section block Toeplitz matrix generated by $g(\lambda)$. Notice that the block Toeplitz matrix A_n is generally not Toeplitz. Our interest is in obtaining the asymptotic distribution of eigenvalues of A_n as $n \to \infty$. The proof is suggested by an argument given in the one-dimensional case ($p=1$) (see [3]) and is based on results in the multidimensional prediction problem [5].

If the real number α is sufficiently small in absolute value $f(\lambda) = [I_p + \alpha g(\lambda)]$ is positive definite for all λ and bounded (I_p is the identity matrix of order p). Let $R_n = I_{np} + \alpha A_n$ be the nth section block Toeplitz matrix generated by $f(\lambda)$. Further denote the (i, j)th block element ($p \times p$ matrix), $i, j = 1, \cdots, n$, of the inverse R_n^{-1} of R_n by ${}_n r_{i,j}^{(-1)}$. The basic result on the determinant of the prediction error covariance matrix in the multidimensional prediction problem [5] tells us that

$$\lim_{n\to\infty} \det ({}_n r_{11}^{(-1)})^{-1} = (2\pi)^p \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log \det\left(\frac{f(\lambda)}{2\pi}\right) d\lambda\right\}$$

since $f(\lambda)/2\pi$ can be regarded as the spectral density function of a p-vector weakly stationary stochastic process. However,

$$\det ({}_n r_{11}^{(-1)})^{-1} = \det (R_n)/\det (R_{n-1}) = \sigma_n^2$$

(see [1, p. 21]). Let $\lambda_{\nu,n}$, $\nu = 1, \cdots, np$, be the eigenvalues of A_n.

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