FOURIER-STIELTJES TRANSFORMS OF MEASURES ON INDEPENDENT SETS

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A subset E of the real line R will be called *independent* if the following is true: for every choice of distinct points $x_1, \cdots, x_k$ in E and of integers $n_1, \cdots, n_k$, not all 0, we have $n_1x_1 + \cdots + n_kx_k \neq 0$. The main result of this note is

THEOREM I. *There exists an independent, compact, perfect set Q in R which carries a positive measure σ whose Fourier-Stieltjes transform*

$$\int_{-\infty}^{\infty} e^{ixy} d\sigma(x) \qquad (y \in R)$$

tends to 0 *as* $|y| \to \infty$.

Sketch of proof. It is known ([5, Theorem IV] and [6, p. 25]) that there is a compact perfect set P in R which is not a basis (i.e., the set of all finite sums $\sum n_i x_i$, with $x_i \in P$ and integers n_i, does not cover R and hence has measure 0) but which carries a positive measure μ whose F.S. transform vanishes at infinity. A certain deformation of P will yield our set Q.

P is constructed as the intersection of a sequence of sets E_r which are unions of 2^r disjoint intervals $I_{j,r}$. Set $P_{j,r} = P \cap I_{j,r}$, for $1 \leqq j \leqq 2^r$.

REMARK 1. Since P is not a basis, the set of all points $w = (w_1, \cdots, w_k)$ in R^k such that $\sum_1^k n_j(x_j + w_j) = 0$ for some choice of $x_1, \cdots, x_k$ in P is, for each choice of integers $n_1, \cdots, n_k$, a closed set of measure 0 (a union of certain hyperplanes).

REMARK 2. Since there exists a function in $L^1(R)$ whose Fourier transform is 1 on $P_{j,r}$ and is 0 on the rest of P, we have

$$\lim_{|y| \to \infty} \int_{P_{j,r}} e^{ixy} d\mu(x) = 0 \qquad (1 \leqq j \leqq 2^r).$$

Choose a sequence $\{c_r\}$, $0 < c_r < 1$, such that $\prod_0^\infty c_r > 0$. Put $f_0(x) = x$, and inductively define a sequence of functions f_r on P, of the form

$$f_r(x) = x + w_{j,r} \qquad (x \in P_{j,r}). \tag{1}$$

Assume f_r is constructed, and has the property that the condition

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