POLYNOMIALS DEFINED BY A DIFFERENCE SYSTEM

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This note is concerned with orthogonal polynomials on the unit circle and their use in probability theory.

Let $f(t) \geqq 0$ (not zero a.e.) be integrable on $-\pi \leqq t \leqq \pi$; then, according to Szegö [1], a system of polynomials $\{\phi_n(z)\}$ orthogonal with respect to $f(t)$ on $-\pi \leqq t \leqq \pi$ are uniquely determined by

(1) (i) $\phi_n(z)$ is a polynomial of degree n in which the coefficient of z^n is real and positive,

(ii) $(1/2\pi)\int_{-\pi}^{\pi} \phi_n(z)\bar{\phi}_m(z)f(t)dt = \delta_{nm}$, $(z = e^{it})$.

Recent results [2; 3; 4] have shown the importance of the Szegö polynomials in discussing fluctuations of sums $S_n = X_1 + \cdots + X_n$, $(n = 0, 1, \cdots)$, of independent, identically distributed random variables X_j. The results derived directly from the theory of the polynomials (1) were necessarily restricted to the case of symmetric, integral-valued random variables. We consider here an alternative definition of the polynomials (1) designed to allow a natural generalization of these results to nonsymmetric, not necessarily discrete-valued random variables. This approach also seems to have connections with prediction theory.

Let $\{\alpha_n\}$ and $\{\beta_n\}$ be given sequences of complex numbers with $\alpha_n\beta_n \neq 1$ for all n, and let u_0 and v_0 be given constants. Then, the system

$$
\begin{aligned}
u_n(z) - u_{n-1}(z) &= \alpha_n z^n v_n(z), \\
v_n(z) - v_{n-1}(z) &= \beta_n z^{-n} u_n(z)
\end{aligned}
\tag{2}
$$

determines polynomials $u_n(z)$ and $v_n(z)$ of at most degree n in z and $1/z$, respectively. The condition $\alpha_n\beta_n \neq 1$ for all n is necessary and sufficient for the existence of $u_n(z)$ and $v_n(z)$ for all n. Let $k_n^2 = \prod_{m=1}^{n} (1 - \alpha_m\beta_m)^{-1}$, and set

$$
\phi_n(z) = z^n v_n(z)/k_n, \qquad \psi_n(z) = z^{-n}u_n(z)/k_n, \tag{3}
$$

where k_n is one of the square roots of k_n^2 (we allow some arbitrariness here). We will connect $\phi_n(z)$ and $\psi_n(z)$ with the Szegö polynomials.

The following notation will be used consistently below. Let $f(t)$ be integrable on $-\pi \leqq t \leqq \pi$ with Fourier coefficients

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