LIFTING THE ACTION OF A GROUP IN A FIBRE BUNDLE

BY T. E. STEWART¹

Communicated by Deane Montgomery, January 8, 1960

1. Suppose that B is a G-space for a given topological group G. That is we are given a continuous map $\bar{\alpha}: G \times B \rightarrow B$ satisfying the equations

$$\bar{\alpha}(u_1 \cdot u_2, b) = \bar{\alpha}(u_1, \bar{\alpha}(u_2, b)), \qquad u_1, u_2 \in G, b \in B,$$

$$\bar{\alpha}(e, b) = b, \qquad e \text{ the identity of } G.$$

Let \mathfrak{B} be a principal bundle over B, [5, p. 35] with total space E and H the structural group so that B may be regarded as the orbit space of E by H. We wish to consider here the problem of putting the two actions together in E in a sense to be made precise below.

2. Let \mathfrak{B} be a bundle with base B and total space E. Suppose B is a G-space given by a function $\overline{\alpha}$ as above. We say that action $(G, \overline{\alpha})$ can be lifted to E in \mathfrak{B} if E can be given the structure of a G-space so that the projection of E onto B in \mathfrak{B} is an equivariant map, i.e. so that if α gives the action of G on E we have the following commutative diagram:

 (G, α) will then be called a lifting of the action $(G, \overline{\alpha})$. A lifting will be called a bundle lifting in \mathfrak{B} if for each $u \in G$ the map $x \to \alpha(u, x)$ of E onto E is a bundle mapping.

For example, a group of diffeomorphisms of a manifold B in the C^r -topology has a bundle lifting in the tangent bundle to B in taking the differential of each element.

PROPOSITION 2.1. If the action $(G, \bar{\alpha})$ on B has a bundle lifting in the principal bundle \mathfrak{B} with structural group H and total space E, then $G \times H$ acts on E in a canonical way. If $(G, \bar{\alpha})$ is a transitive action so is the action of $G \times H$. If the action $(G, \bar{\alpha})$ is free so is that of $G \times H$.

If (G, α) is the bundle lifting in B of $(G, \overline{\alpha})$ define the action $(G \times H, \beta)$ in E by $\beta((u, h), x) = \alpha(u, x) \cdot h$.

¹ The author holds a National Science Foundation Postdoctoral Fellowship.