REMARKS ON AFFINE SEMIGROUPS[1]

BY A. D. WALLACE

Communicated by Einar Hille, December 28, 1959

A *semigroup* is a nonvoid Hausdorff space together with a continuous associative multiplication, denoted by juxtaposition. In what follows S will denote one such and it will be assumed that S is *compact*. It thus entails no loss of generality to suppose that S is contained in a locally convex linear topological space $\mathfrak{X}$, but no particular imbedding is assumed. For general notions about semigroups we refer to [**3**] and for information concerning linear spaces to [**2**].

It has been known for some time [**3**] that if $\mathfrak{X}$ is finite dimensional, if S is convex (recall that S is compact) and if S has a unit (always denoted by u) then the maximal subgroup, H_u, which contains u is a subset of the boundary of S relative to $\mathfrak{X}$.

Let F denote the boundary of S, K the minimal ideal of S and, for any subset A of S, let

$$P(A) = \{x \mid x \in S \text{ and } xA = A\}.$$

As is customary, AB denotes the set of all products ab with $a \in A$ and $b \in B$ and we generally write x in place of $\{x\}$. It will be convenient to abbreviate $P(S)$ by P. The structure of P is known in the following sense—supposing that $P \neq \square$ the set $P \cap E \neq \square$ and is indeed the set of left units of S, E being the set of idempotents. Moreover, if $e \in P \cap E$ then Pe is a maximal subgroup of S and the assignment $(x, y) \rightarrow xy$ is an iseomorphism (topological isomorphism) of $Pe \times (P \cap E)$ onto P. The following is a corollary to the principal result of [**4**]:

THEOREM 1. *If S is compact and convex and if $S \neq K$ then*

$$P(F) = P(S) \subset F.$$

It should be noticed that if S has a unit then $P = H_u$.

The quantifier *affine* will be applied to S if S is convex and if also $x(ty + (1-t)z) = txy + (1-t)xz$ and $(ty + (1-t)z)x = tyx + (1-t)zx$ for any x, y and $z \in S$ and any t with $0 \leqq t \leqq 1$. This differs a little from the definition in [**1**].

This is a particularly pleasant concept because of its generality and because of the host of examples of a simple geometric character. One such is the convex hull of the n roots of unity, using complex

[1] This work was supported by a grant from the National Science Foundation.