SIMULTANEOUS UNIFORMIZATION[1]

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We shall show that any *two* Riemann surfaces satisfying a certain condition, for instance, any two closed surfaces of the same genus $g>1$, can be uniformized by *one* group of fractional linear transformations (Theorem 1). This leads, in conjunction with previous results [2; 3], to the simultaneous uniformization of *all* algebraic curves of a given genus (Theorems 2–4). Theorem 5 contains an application to infinitely dimensional Teichmüller spaces.

1. Let S be an abstract Riemann surface, f a homeomorphism of bounded eccentricity of S onto another such surface S', and $[f]$ the homotopy class of f. We call $(S, [f], S')$ a *coupled pair* of Riemann surfaces, an *even* (*odd*) pair if f preserves (reverses) orientation. Two coupled pairs, $(S, [f], S')$ and $(S_1, [f_1], S')$ are called *equivalent* if there exist conformal homeomorphisms h and h' with $h(S)=S_1$, $h'(S) = S_1'$, and $[h'fh^{-1}]=[f_1]$.

EXAMPLE. Let m be a Beltrami differential on the Riemann surface S_0, i.e. a differential of type $(-1, 1)$, $m=(\zeta)d\bar{\zeta}/d\zeta$, with $|\mu| \leqq \text{const.}<1$. By S_0^m we denote the surface S_0 with the conformal structure redefined by means of the local metric $|d\zeta+\mu d\bar{\zeta}|$. With m there is associated the even pair $(S_0^m, [1], S)$, where 1 is the identity mapping, and the odd pair $(S_0^m, [\iota], \bar{S}_0)$ where ι denotes the natural mapping of S_0 onto its mirror image $\bar{S}_0$. The latter is defined by replacing each local uniformization ζ on S_0 by $\bar{\zeta}$.

A group G of Möbius transformations will be called *quasi-Fuchsian* if there exists an oriented Jordan curve γ_G (on the Riemann sphere P) which is fixed under G, and if G is fixed-point-free and properly discontinuous in the domains $I(\gamma_G)$ and $E(\gamma_G)$ interior and exterior to γ_G, respectively. If γ_G is a circle, G is a Fuchsian group.

A quasi-Fuchsian group G is canonically isomorphic to the fundamental groups of the two Riemann surfaces $S_1=I(\gamma_G)/G$ and $S_2 = E(\gamma_G)/G$, modulo inner automorphisms. If the resulting isomorphisms of the fundamental groups of S_1 onto those of S_2 can be induced by an orientation reversing homeomorphism f of bounded eccentricity, G is called *proper*. In this case $[f]$ is uniquely determined.

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