THE SCOPE OF THE ENERGY METHOD[1]

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In this paper the energy method is applied to show the *stability* of a certain class of difference approximations to linear hyperbolic partial differential equations with variable coefficients. We shall consider first order hyperbolic systems:

$$u_t = A u_x, \tag{1}$$

u a vector, A a coefficient matrix with real and distinct eigenvalues. We approximate this by explicit, one level difference schemes, i.e., of the form

$$v_k = \sum_{-N}^{N} c_j u_{k-j} \tag{2}$$

where u_j and v_j denote the values of the approximate solution at position $j\Delta$ and at times t and $t+\Delta$ respectively. We have chosen here for simplicity the same value Δ for both time and space increments.

The coefficients c_j depend on the index k; we assume that this dependence is Lipschitz continuous, i.e., that $c(k+1)-c(k)=O(\Delta)$.

Since the differential equation (1) is homogeneous, $u=$const. is a solution. We assume, as part of the consistency of (1) and (2), that $u=$const. also is a solution of (2), i.e., that

$$\sum c_j = I. \tag{3}$$

A difference scheme is called *stable* in the L_2-norm if the L_2-norm of any solution at any given time $t=T$ remains *uniformly bounded*, as Δ tends to zero, by a constant multiple of its L_2-norm at $t=0$. It is well known, see [4], that if (2) is formally consistent with (1) then solutions of (2) approach solutions of (1) as Δ tends to zero if and only if (2) is stable.

If the coefficients of (2) are constant, then the question of stability can be easily settled by Fourier analysis.

Put

$$V(\theta) = \sum v_k e^{ik\theta}, \qquad U(\theta) = \sum u_j e^{ij\theta};$$

then it follows from (2) that

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