RESEARCH PROBLEMS

1. Richard Bellman: *Uniform approximation of roots.*

Let $f(x)$ be a monotone increasing function of x with positive continuous derivative for $x \geqq 0$, with $f(0)=0$, $f(\infty)=\infty$. Consider the equation

$$f(x) = y, \tag{1}$$

possessing the unique solution $x=f^{-1}(y)$ for $y \geqq 0$. Let

$$x_{n+1} = x_n + \frac{y - f(x_n)}{f'(x_n)}, \qquad x_0 = z, \tag{2}$$

be the sequence of successive approximations to $f^{-1}(y)$ furnished by Newton's method.

Determine $z=z(a, b, n)$ so that

$$\operatorname*{Max}_{a \leqq y \leqq b} \left| x_n - f^{-1}(y) \right| \tag{3}$$

is a minimum, where $0<a<b<\infty$, and determine the asymptotic behavior of $z(a, b, n)$ as $n \to \infty$.

For $f(x)=x^2$, it is known that $z(a, b, n) \to (ab)^{1/4}$ as $n \to \infty$. (Received November 26, 1956.)

2. Richard Bellman: *Maximization of linear functions.*

At the present time, there is no systematic technique for solving the problem of maximizing the linear form $L(x)=\sum_{i=1}^{N} a_i x_i$ subject to the constraints $\sum_{j=1}^{N} b_{ij} x_j \leqq c_i$, $i=1, 2, \cdots, M$, where the a_i and b_{ij} are positive integers, or zero, and the x_j are constrained to be positive integers or zero. On the other hand, if this constraint on integral solutions is removed, the solution is readily obtained for small M, and there exist effective algorithms for large M.

For the case $M=1$, let $f_N(c_1)$ denote the maximum of $L(x)$ under integral constraints and $g_N(c_1)$ denote the solution under the constraint $x_i \geqq 0$. Define the function

$$\phi(N) = \operatorname*{Sup}_{a_i, b_{1j} \geqq 1} \left[\operatorname*{Sup}_{c \geqq \operatorname{Min}_j b_{1j}} \frac{g_N(c)}{f_N(c)} \right].$$

What is the order of magnitude of $\phi(N)$ as $N \to \infty$, and in particular, is it bounded?

Consider the corresponding problem for general M where

$$\phi_M(N) = \operatorname*{Sup}_{a_i, b_{ij} \geqq 1} \left[\operatorname*{Sup}_{c_i \operatorname{Min}_i b_{ij}} \frac{g_N(c_1, c_2, \cdots, c_M)}{f_N(c_1, c_2, \cdots, c_M)} \right]$$

(Received October 3, 1956.)