ON SMOOTHING OPERATIONS AND THEIR GENERATING FUNCTIONS[1]

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INTRODUCTION

In this paper we are mainly concerned with two kinds of linear transformations, the sequence convolution transformation

$$(1) \qquad y_n = \sum_{\nu=-\infty}^{\infty} a_{n-\nu} x_\nu$$

and the integral convolution transformation

$$(2) \qquad g(x) = \int_{-\infty}^{\infty} \Lambda(x - t) f(t) dt,$$

where the sequence $\{a_n\}$ and the function $\Lambda(x)$ are thought of as given. In §2 we also consider the ordinary linear transformation

$$(3) \qquad y_i = \sum_{k=1}^{n} a_{ik} x_k \qquad (i = 1, \cdots, m).$$

The loosely connected topics to be discussed concerning these transformations are perhaps best brought together under the general subject of smoothing operations.

In §1 we are concerned with transformations (1) of the kind used for the purpose of smoothing numerical data. Erastus L. De Forest proposed long ago the problem of describing the asymptotic behavior of the coefficients of high-order iterates of (1) (see Wolfenden [**15**]). This question, as well as the question of when a formula (1) may rightly be called a smoothing formula, was answered by the author in [**38**] and [**31**]. It is shown in §1 that the author's criterion for a smoothing formula is essentially of the nature of a stability condition of the kind required of difference methods for the numerical integration of partial differential equations. From this point of view De Forest's problem amounts to constructing by difference methods the fundamental solutions of certain parabolic differential equations. These remarks are merely special cases of Fritz John's recent work

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