LIE THEORY OF SEMI-GROUPS OF LINEAR TRANSFORMATIONS

EINAR HILLE

1. **Introduction.** In the Colloquium Lectures which I had the honor of delivering to the Society at the Wellesley meeting in August 1944, an outline was given of a theory of *one-parameter semi-groups of linear bounded operators on a complex (B)-space $\mathfrak{X}$ to itself.* The problem here is the study of a family of linear bounded transformations $\mathfrak{S}=\{T(\alpha)\}$, defined for $\alpha>0$, with the product law

$$T(\alpha)T(\beta) = T(\alpha+\beta). \tag{1.1}$$

Such families arise in the most varied branches of classical and of modern analysis and are interesting for their own sake as well as for the many applications.

An extension to the n-parameter case was presented to the Society in October 1944 (abstract 51-1-15). Here the parameter $a=(\alpha_1, \alpha_2, \cdots, \alpha_n)$ is a vector in n-dimensional real euclidean space E_n, the operators $T(a)$ are defined for non-negative values of the components of a, and the *product law* reads

$$T(a)T(b) = T(a+b) \tag{1.2}$$

with $a+b=(\alpha_1+\beta_1, \cdots, \alpha_n+\beta_n)$. These operators commute. If $\|T(a)\|$ is bounded for small a, if certain unions of range spaces $T(a)[\mathfrak{X}]$ are dense in the space $\mathfrak{X}$, and if $T(a)$ is a strongly measurable function of a, then $T(a)$ is actually strongly continuous for all a and $T(h)x\to x$ for each x when $h\to 0$. Further $T(a)$ is the direct product of n commuting one-parameter semi-groups

$$T(a) = T_1(\alpha_1)T_2(\alpha_2)\cdots T_n(\alpha_n). \tag{1.3}$$

It turned out later that the analysis could be extended, at least in part, to the case in which the parameter set is an *open positive cone* $\mathfrak{C}$ in a (B)-space $\mathfrak{P}$. Here $\mathfrak{C}$ is an open set, if a and b are in $\mathfrak{C}$ so are $\alpha a+\beta b$ for $0\leqq\alpha$, $0\leqq\beta$, $0<\alpha+\beta$. The product law is still given by (1.2).

These investigations with many extensions and numerous applications have now appeared in book form ([6] in the References at the end of this address). The earliest results on continuity in the one-parameter case are due to N. Dunford [2] and extensions to the

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