THE SCHWARZIAN DERIVATIVE AND SCHLICHT FUNCTIONS

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It is customary to formulate the inequalities of the "Verzerrungssatz" type for analytic functions $w=f(z)$, schlicht in the unit circle, with reference to a specific normalization. The two normalizations mainly used are: (a) $f(z)$ is finite in $|z|<1$, $f(0)=0$, $f'(0)=1$; (b) $f(z)$ has a pole at $z=0$ with the residue 1. If we want to obtain inequalities which are independent of any particular normalization, we have to use quantities which are invariant with regard to an arbitrary linear transformation of the z-plane. The simplest quantity of this type is the Schwarzian differential parameter

$$\{w, z\} = \left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2, \tag{1}$$

also called the Schwarzian derivative of w with regard to z.

It is easy to obtain an upper bound for $\{w, z\}$ by a simple transformation of the classical inequality $|a_1| \leqq 1$ valid for functions $w=f(z)=z^{-1}+a_0+a_1z+ \cdots$ schlicht in the unit circle. Indeed, applying this inequality to the coefficient of z in the expansion of the schlicht function

$$g(z) = \frac{f'(x)(1-|x|^2)}{f((x+z)/(1+\bar{x}z)) - f(x)} = \frac{1}{z} + \bar{x} - \frac{1}{2}\frac{f''(x)}{f'(x)}(1-|x|^2)$$
$$- \frac{1}{6}(1-|x|^2)^2\left[\left(\frac{f''(x)}{f'(x)}\right)' - \frac{1}{2}\left(\frac{f''(x)}{f'(x)}\right)^2\right]z + \cdots$$
$$(|x| < 1),$$

we obtain $|\{w, z\}| \leqq 6(1-|z|^2)^{-2}$.

We shall now show that by replacing the number 6 in this inequality by 2, this necessary condition for the schlichtness of $f(z)$ in $|z|<1$ becomes sufficient.

THEOREM I. *In order that the analytic function $w=f(z)$ be schlicht in $|z|<1$, it is necessary that*

$$|\{w, z\}| \leqq \frac{6}{(1-|z|^2)^2}$$

Received by the editors May 24, 1948.