A RATIO TEST

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The d'Alembert ratio test has numerous extensions that are effective in some cases where the d'Alembert test fails; that is, when the limit of the ratio a_n/a_{n-1} is 1. Examples are Raabe's test and Gauss' test. The following test is simpler than any of these, and is easy to prove and to remember.

THEOREM. *A series of positive terms* (1) $\sum a_n$ *converges if* $\lim_{n\to\infty} (a_n/a_{n-1})^n < 1/e$, *and diverges if* $\lim_{n\to\infty}(a_n/a_{n-1})^n > 1/e$.

More generally, (1) *converges if* $\lim\sup\, (a_n/a_{n-1})^n < 1/e$. It diverges if $(a_n/a_{n-1})^n \geqq 1/e$ *for all* n *sufficiently large. In particular* (1) *diverges if* $\lim\inf\, (a_n/a_{n-1})^n > 1/e$.

COROLLARY. *A series of real or complex terms* $\sum a_n$ *converges absolutely if* $\lim\sup\, |a_n/a_{n-1}|^n < 1/e$.

The proof is by the comparison ratio test with $\sum n^{-s}$ as the comparison series. To prove the convergence part of the test, suppose $\lim\sup\, (a_n/a_{n-1})^n = e^{-d} < e^{-1}$. Then $d > 1$. Consider the series (2) $\sum b_n = \sum n^{-s}$ with $1 < s < d$. Then we have $\lim\, (b_n/b_{n-1})^n = \lim\, (1-1/n)^{ns} = e^{-s}$. Since $e^{-d} < e^{-s}$, then ultimately $(a_n/a_{n-1})^n < (b_n/b_{n-1})^n$, and therefore $a_n/a_{n-1} < b_n/b_{n-1}$. It follows that (1) converges, since (2) does.

To prove the divergence part of the test, suppose $(a_n/a_{n-1})^n \geqq 1/e$ for n sufficiently large. For the harmonic series (3) $\sum c_n = \sum n^{-1}$, we have $(c_n/c_{n-1})^n = (1-1/n)^n < e^{-1}$ for all n. Hence ultimately $(a_n/a_{n-1})^n > (c_n/c_{n-1})^n$, and therefore $a_n/a_{n-1} > c_n/c_{n-1}$. Since (3) diverges, so does (1). This completes the proof. In the same manner one can prove the following generalization.

THEOREM. *A series of positive terms* (1) $\sum a_n$ *converges if* $\lim\sup\, (a_n/a_{n-k})^n < 1/e^k$, *and diverges if* $(a_n/a_{n-k})^n \geqq 1/e^k$ *for all* n *sufficiently large.*

As an application, consider the Dirichlet series (4) $\sum a_n n^{-s}$, where $s = \sigma + i\tau$. Suppose $\lim\, |a_n/a_{n-1}|^n = e^{-d}$. Then it follows from the test that the series (4) converges absolutely at every point of the half-plane $\sigma > 1-d$, and converges absolutely at no point of the half-plane $\sigma < 1-d$.

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