EXISTENCE THEOREMS CONNECTED WITH THE PICARD-VESSIOT THEORY OF HOMOGENEOUS LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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1. Introduction. The Picard-Vessiot theory, as recently reformulated by the author,[1] deals with an abstract ordinary differential field $\mathcal{F}$ of characteristic 0 having an algebraically closed field of constants $\mathcal{C}$, and a differential extension field $\mathcal{G}$ over $\mathcal{F}$ with the two properties:

(a) There exists a homogeneous linear differential polynomial $L(y)=y^{(n)}+p_1y^{(n-1)}+\cdots+p_ny$ (each p_i in $\mathcal{F}$) which has a fundamental system of solutions $\eta_1, \cdots, \eta_n$ such that $\mathcal{G}=\mathcal{F}\langle\eta_1, \cdots, \eta_n\rangle$;[2]

(b) The field of constants of $\mathcal{G}$ is $\mathcal{C}$.

Such a $\mathcal{G}$ is called a *Picard-Vessiot* extension of $\mathcal{F}$. It is to be noted that the extension $\mathcal{G}$ is given, and the existence of the differential polynomial $L(y)$ with the properties (a) and (b) is postulated. It is not immediately apparent, and it would be of interest to know, whether a given $L(y)$, with coefficients p_i in $\mathcal{F}$, always has a fundamental system of solutions $\eta_1, \cdots, \eta_n$ such that $\mathcal{F}\langle\eta_1, \cdots, \eta_n\rangle$ is a Picard-Vessiot extension of $\mathcal{F}$ (that is, contains no constant not in $\mathcal{C}$). This question was posed by R. Baer (in his critical note on the then current status of the Picard-Vessiot theory, included among comments by O. Haupt in F. Klein's *Vorlesungen über hypergeometrische Funktionen*, Berlin, 1933), who remarked that the difficulty lay not in proving the existence of a fundamental system of solutions (see PV, §15), but in proving the existence of one which brings in no new constants.

A differential extension field $\mathcal{H}$ of $\mathcal{F}$ may be an extension of $\mathcal{F}$ by integrals, exponentials of integrals, and algebraic functions. If it is, and if the field of constants of $\mathcal{H}$ is still $\mathcal{C}$, then $\mathcal{H}$ is called a *liouvillian* extension of $\mathcal{F}$. The Picard-Vessiot theory provides a group-theoretic answer to the question of when a Picard-Vessiot extension $\mathcal{G}$ of $\mathcal{F}$ is

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[1] *Algebraic matric groups and the Picard-Vessiot theory of homogeneous linear ordinary differential equations*, Ann. of Math. (2) vol. 49 (1948) pp. 1–42. This paper, referred to below as "PV", contains the necessary background for the present note.

[2] The notation $\mathcal{F}\langle\cdots\rangle$ indicates, as usual, differential field adjunction. Thus $\mathcal{F}\langle\eta_1, \cdots, \eta_n\rangle$ is the differential field consisting of all differential rational functions of $\eta_1, \cdots, \eta_n$ with coefficients in $\mathcal{F}$.