POLYNOMIALS IN TOPOLOGICAL FIELDS

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1. **Introduction.** Let F be a real closed field in the sense of Artin-Schreier [1, 8],[1] and $f(x)$ a polynomial with coefficients in F. It is known that $f(x)$ attains a maximum and minimum in any interval $a \leqq x \leqq b$. Recently, Habicht [4] has begun the study of polynomials in several variables over a real closed field and proved the following theorem: if $f(x_1, \cdots, x_n)$ is positive for $-m \leqq x_i \leqq m$, then f has a positive lower bound in this region. An equivalent statement is that f maps the region into a closed subset of F, where we give F its order topology (the open intervals are a base for the open sets). This latter formulation suggests possible extensions of Habicht's theorem to more general topological fields.

In this note we shall examine such extensions. In §3 we obtain quite complete results for polynomials in one variable over fields of "type V" (§2). But simple examples show that for two or more variables the situation is more complicated. We do however obtain a result (Theorem 4) which immediately implies Habicht's theorem and which is valid for a somewhat wider class of topological fields than real closed fields in their order topology. A further result for polynomials in two variables appears in Theorem 3.

2. **Fields of type** V. We begin by recalling some definitions. In a topological ring A we call a set B right bounded if for any neighborhood U of 0 there exists a neighborhood V of 0 such that $BV \subset U$. Left boundedness is analogously defined and a set is bounded if it is both right and left bounded. We denote by A_n the n-dimensional vector space over A (Cartesian product topology), and a subset of A_n is bounded if it is bounded coordinate-wise. Any compact set is bounded, and in many arguments such as the following, bounded sets behave somewhat like compact ones.

LEMMA 1. *Let f, g be functions defined on a subset S of A_n and taking values in A. Suppose f, g are bounded and uniformly continuous on S. Then $f+g$ and fg are bounded and uniformly continuous on S.*

The proof is obvious except perhaps for the uniform continuity of fg, which depends upon iterated use of the identity

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[1] Numbers in brackets refer to the bibliography at the end of the paper.