UNIVERSAL QUATERNARY QUADRATIC FORMS

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Let the positive quaternary quadratic form $F=\sum_{i,j=1}^{4} b_{ij}x_ix_j$, where b_{ij} are integers and $b_{ij}=b_{ji}$, have its hessian equal to H'. Then it can be shown that every positive quaternary quadratic form whose hessian is H' is equivalent to a form F in which $b_{11}\leqq(4H')^{1/4}$, $2|b_{ij}|\leqq b_{11}$, $j=2, 3, 4$, and $b_{11}F-(\sum_{j=1}^{4} b_{1j}x_j)^2$ is a reduced positive ternary quadratic form $T=\sum_{i,j=2}^{4}B_{ij}x_ix_j$ whose hessian is b_{11}^2H'. Further if $b_{11}=2$, we may take $b_{1j}\geqq 0$, $B_{23}\geqq 0$ and $B_{22}B_{34}-B_{23}B_{24}\geqq 0$, and in case $B_{23}=0$ we may choose $B_{24}\geqq 0$. A form satisfying these conditions is a reduced form.

This paper is concerned with the determination of the positive forms $f_0=\sum_{i,j=1}^{4} c_{ij}x_ix_j$, in which $c_{ij}=c_{ji}$, c_{ii} and $2c_{ij}$ are integers, which are universal. For the case in which $c_{ij}=0$ if $i\neq j$ the forms which represent all positive integers have been determined by Ramanujan.[1] L. E. Dickson[2] considered those forms in which $c_{1j}=0$, $j=2, 3, 4$. Since the minimum positive integer which is represented by f_0 is 1, this form is equivalent to a form f whose first coefficient is 1. Let $F=2f=\sum_{i,j=1}^{4} a_{ij}x_ix_j$, where $a_{ij}=a_{ji}$ are integers, $a_{11}=2$ and $a_{ii}\equiv 0 \pmod 2$, be a reduced form whose hessian is H. Then $4f=X^2+T$ where $X=\sum_{j=1}^{4} a_{1j}x_j$ and $T=\sum_{i,j=2}^{4} A_{ij}x_ix_j$ $(A_{ij}=a_{11}a_{ij}-a_{1i}a_{1j})$ is a reduced positive ternary quadratic form whose hessian is $4H$. Since f is universal, $4f$ represents all positive multiples of 4. The minimum positive integer M which is represented by the ternary T is $\leqq 8$ and $\equiv 0$ or $3 \pmod 4$. Hence $M=3, 4, 7$ or 8.

The ternary form[3] $MT=U^2+Q$ where $U=Mx_2+A_{23}x_3+A_{24}x_4$, and $Q=Ax_3^2+2Bx_3x_4+Cx_4^2$ is a reduced binary quadratic form whose hessian is $4MH$, and where $A=MA_{33}-A_{23}^2$, $B=MA_{34}-A_{23}A_{24}$ and $C=MA_{44}-A_{24}^2$, while $2|A_{23}|$ and $2|A_{24}|$ are each $\leqq M$. We may choose $A_{23}\geqq 0$ and $B\geqq 0$, while if $A_{23}=0$, we may take $A_{24}\geqq 0$.

The least positive integer A which is represented by Q if $M=3$ is $\leqq 24$ and $\equiv 0$ or $8 \pmod{24}$. If $M=4$, $A\leqq 48$ and $\equiv 0$ or $12 \pmod{16}$ if $A_{23}=0$, and $A\equiv 12$ or $8 \pmod{16}$ if $A_{23}=2$. For $M=7$, $A\leqq 84$ and $\equiv 0, -4, -8$, or $-16 \pmod{28}$ according as $A_{23}=0, 1, 2$ or 3. Finally

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[1] Proc. Cambridge Philos. Soc. vol. 19 (1916–1919) pp. 11–21.

[2] Amer. J. Math. vol. 49 (1927) pp. 39–56.

[3] Landau, *Handbuch* · · · *Verteilung der Primzahlen*, 1909, p. 545.