ON AN INEQUALITY OF P. TURÁN CONCERNING LEGENDRE POLYNOMIALS

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The following remarkable inequality is due to the Hungarian mathematician P. Turán: If $P_n(x)$ denotes as usual Legendre's polynomial of the nth degree, we have

$$(1)\qquad \Delta_n(x) = (P_n(x))^2 - P_{n-1}(x)P_{n+1}(x) \geqq 0, \qquad n \geqq 1;\ -1 \leqq x \leqq 1,$$

with equality only for $x = \pm 1$. The purpose of this note is to give several proofs for this theorem different from that of Turán.[1]

1. Proof. The following arrangement is somewhat similar to that of Turán. By using the classical recursion

$$(2)\qquad P_{n+1}(x) = \frac{2n+1}{n+1} xP_n(x) - \frac{n}{n+1} P_{n-1}(x)$$

we find for the polynomial $\Delta_n(x)$ the representation

$$(3)\qquad P_n^2 + \frac{n}{n+1} P_{n-1}^2 - \frac{2n+1}{n+1} xP_nP_{n-1}.$$

This is a quadratic form in P_n and P_{n-1} which is positive provided

$$(4)\qquad \frac{n}{n+1} > \left(\frac{n+1/2}{n+1} x\right)^2, \quad \text{or} \quad |x| < \frac{(n(n+1))^{1/2}}{n+1/2} = \cos\theta_0.$$

For these x the theorem is already proved. For the remaining $x = \cos\theta$, that is, for $0 < \theta \leqq \theta_0$, we use Mehler's formula

$$(5)\qquad P_n(\cos\theta) = \frac{2}{\pi}\int_0^\theta \frac{\cos(n+1/2)u}{(2(\cos u - \cos\theta))^{1/2}} du$$

and obtain

$$(6)\qquad \begin{aligned} \Delta_n(\cos\theta) &= \pi^{-2}\int_0^\theta\int_0^\theta (\cos u - \cos\theta)^{-1/2}(\cos v - \cos\theta)^{-1/2} \\ &\cdot \{2\cos(n+1/2)u\cos(n+1/2)v - \cos(n-1/2)u\cos(n+3/2)v \\ &\qquad - \cos(n-1/2)v\cos(n+3/2)u\}\, dudv. \end{aligned}$$

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[1] I owe Mr. Turán also some other remarkable properties of the polynomial $\Delta_n(x)$.