A BOUND FOR THE ERROR IN COMPUTING THE BESSEL FUNCTIONS OF THE FIRST KIND BY RECURRENCE

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It is sometimes useful to obtain the value of a Bessel function of the first kind of a fairly high integral order. Inasmuch as $J_0(x)$ and $J_1(x)$ are relatively easy to compute, even apart from the existence of numerous tables of $J_0(x)$ and $J_1(x)$ to considerable accuracy, one obvious method of attack is to use the recurrence relation,

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x), \tag{1}$$

the starting values being the already available values of $J_0(x)$ and $J_1(x)$. In using this formula, however, it is necessary to have a bound for the error in $J_n(x)$ caused by errors in $J_0(x)$ and $J_1(x)$. It is the purpose of this note to furnish such a bound.

Let $e_n(x)$ represent the difference between the value of $J_n(x)$ and the approximate value obtained by applying (1) to values which differ from $J_0(x)$ and $J_1(x)$ by errors $e_0(x)$ and $e_1(x)$, respectively. Since $e_n(x)$ is additive to the approximate value and since both the approximate and true values satisfy the recurrence formula, $e_n(x)$ must also satisfy the recurrence formula.

If exact values of $e_0(x)$ and $e_1(x)$ were known, an exact value of $e_n(x)$ could be obtained by the recurrence relation (1). However, by the nature of the problem bounds for the absolute values of $e_0(x)$ and $e_1(x)$, and in some cases their signs, are the only information available. These bounds cannot be used as starting values for recurrence to find a bound to the absolute value of $e_n(x)$ since in the relation (1) a substitution of the bounds on the right-hand side may yield a result whose absolute value is well within the bound for the absolute value of the left-hand side.

In the relation due to Lommel and Hankel,[1]

$$2/\pi x = Y_n(x)J_{n+1}(x) - Y_{n+1}(x)J_n(x), \tag{2}$$

$Y_n(x)$ being the customary Bessel function of the second kind of order n, setting $n=0$, multiplying both sides by $(\pi x/2)e_0(x)$, adding the null term $(\pi x/2)e_1(x)Y_0(x)J_0(x) - (\pi x/2)e_1(x)Y_0(x)J_0(x)$ and rearranging the terms give

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[1] Watson, G. N., *A treatise on the theory of Bessel functions*, 2d ed., 1944, p. 77.