A PROOF OF TWO FUNDAMENTAL THEOREMS ON LINEAR TRANSFORMATIONS IN HILBERT SPACE, WITHOUT USE OF THE AXIOM OF CHOICE

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The definitions and notations used in this paper may be found in the volume *Linear transformations in Hilbert space* by M. H. Stone,[1] which work I shall quote as S.

Many of the consequences of S, chaps. 4–9, are based on the existence of transformations of a particular kind called projections (S, Definition 2.16), and such existence is a consequence of S, Theorem 1.23, whose proof is based on S, Theorem 1.18. The last one is a particular case of a Hausdorff theorem[2] on abstract spaces, and its proof depends upon the axiom of choice. I shall show, without use of that axiom, that "every closed linear manifold $\mathfrak{M}$ in the Hilbert space $\mathfrak{H}$ has an orthogonal complement $\mathfrak{H}\ominus\mathfrak{M}$." The consistency of S, Definition 2.16 will thus be ensured.

Another theorem based on the axiom of choice is S, Theorem 2.25. This one I shall prove as well without the aid of the axiom.

It is necessary to make clear the meaning of the word "closed" applied to a subset $\mathfrak{M}$ of the Hilbert space $\mathfrak{H}$.

Let f be a point of $\mathfrak{H}$. f is said to be a *point of accumulation* of $\mathfrak{M}$ if for every real number $\epsilon>0$ there exists a point $g\neq f$ of $\mathfrak{M}$ such that $|g-f|<\epsilon$. $\mathfrak{M}$ is said to be *closed* when every point of accumulation of $\mathfrak{M}$ is a point of $\mathfrak{M}$.

An alternative definition is the following: f is a point of accumulation of $\mathfrak{M}$ if there exists an infinite sequence $\{g_i\}$ of points of $\mathfrak{M}$ such that $g_i\neq f$ $(i=1, 2, \cdots)$, $\lim_{i\to\infty} g_i=f$; $\mathfrak{M}$ is said to be closed when every point of accumulation of $\mathfrak{M}$ is a point of $\mathfrak{M}$.

It is well known that, if the axiom of choice is accepted, these two definitions are equivalent; if not, a subset $\mathfrak{M}$ closed according to the first definition is also closed according to the second one, but the converse is not necessarily true. Therefore, we shall assume the first meaning of the word "closed" to be the correct one.

In the logical development of the theory, Theorems 1.3 to 1.14 of S can be stated and proved: it must be remarked that in the proof of S, Theorem 1.13, a well determined orthonormal set $\{\phi_n\}$ is found. Therefore, when a sequence $\{f_n\}$ is given, it is possible to

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[1] Amer. Math. Soc. Colloquium Publications, vol. 15, 1932.

[2] Hausdorff, *Grundzüge der Mengenlehre*, Theorem VIII, p. 273.