A GENERALIZATION OF STEINER'S FORMULAE

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Let C be an arbitrary convex curve in the plane of length L and area F; and let C_ρ be a curve parallel to C at a distance ρ from it, of length L_ρ and area F_ρ. Then according to Steiner's classical result:

$$L_\rho = L + 2\pi\rho, \qquad F_\rho = F + \rho L + \pi\rho^2.$$

In this paper we develop a generalization of these formulae for curves lying on a curved surface whose curvature $K(v^1, v^2)$ (referred to geodesic parallel coordinates) is a function of v^2 alone. Explicit formulae are derived in the case of surfaces of constant curvature. In this treatment it is necessary to put certain restrictions on the curve C and the distance ρ to replace Steiner's assumption of convexity. These restrictions (which are discussed below) are stated in their most obvious form, and a discussion of methods of relaxing them is deferred to a later paper. Our chief results are contained in the formulae (12) and (15) below.

Let the curve C be a simple, closed, bounding, and differentiable curve on the surface S. Choose a coordinate system in which $v^1=0$ is the curve C, and in which $v^2=$constant are the geodesics orthogonal to C. Further let v^2 be the arc length of C measured positively for motion on the curve which keeps the bounded area to the left, and let v^1 be the arc length of geodesics normal to C measured positively outward from C. Choose the unit normals to C so that they point toward the interior of C. Then we have:

$$ds^2 = (dv^1)^2 + g_{22}(v^1, v^2)(dv^2)^2; \qquad g_{22}(0, v^2) = 1. \tag{1}$$

For the moment we ignore the question of determining the region of S within which such a coordinate system is valid, and proceed to compute $(g_{22})^{1/2}$. In this coordinate system we have the following relations (see L. P. Eisenhart *An introduction to differential geometry*, pp. 181 and 188)

$$\frac{\partial^2 (g_{22})^{1/2}}{\partial v^1 \partial v^1} + K(g_{22})^{1/2} = 0, \tag{2}$$

$$\kappa_g(v^2) = \left[\frac{\partial (g_{22})^{1/2}}{\partial v'}\right]_{v^1=0}, \tag{3}$$

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