TERNARY BOOLEAN ALGEBRA

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1. Introduction. The present paper[1] is concerned with a ternary operation in Boolean algebra. We assume a degree of familiarity with the latter [1, 2],[2] and by the former we shall mean simply a function of three variables defined for elements of a set K whose values are also in K. Ternary operations have been discussed in groupoids [4] and groups [3]; in Boolean algebra an operation different from the one introduced here was discussed by Whiteman [5].

By a simple set of postulates (§2), we define a ternary system, which we call a *ternary Boolean algebra*, from which Boolean algebras are obtained by specialization of the ternary operation, and which itself may be considered as a more general binary system with as many binary operations as elements, each having the properties of the Boolean operations (§4). The ternary Boolean algebra is homogeneous and there is a one-to-one correspondence between distinct ternary algebras and abstract Boolean algebras (§5, §7); thus the ternary algebra provides a new postulational approach to Boolean algebra. The ternary operation has a unique realization in Boolean algebra (§6). Other applications of ternary operations and the matter of a valid representation for ternary Boolean algebra are left to a subsequent paper.

2. Postulates for ternary Boolean algebra. Let K be a system consisting of a set of elements $a, b, \cdots$, and two operations under which the system is closed, one ternary, a^bc, and the other unitary a'. These satisfy the following relations for all a, b, c, d, and e:

$$a^b(c^de) = (a^bc)^d(a^be), \tag{2.1}$$

$$a^bb = b^ba = b, \tag{2.2}$$

$$a^bb' = b'^ba = a. \tag{2.3}$$

The system thus defined we shall call a *ternary Boolean algebra*.

It is easily verified that the following function in Boolean algebra satisfies the postulates:

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[2] Numbers in brackets refer to the bibliography at the end of the paper.