TWO-DIMENSIONAL GEOMETRIES WITH ELEMENTARY AREAS

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Whereas area in spaces with a smooth Riemann metric has been widely studied, very little is known regarding area in spaces with general metrics. It is natural to ask, first, in which general spaces the most familiar types of formulas for area hold.

The present note answers this question in two cases for two-dimensional spaces in which the geodesic connection is locally unique.[1] It shows under very weak assumptions regarding the nature of area:

I. *If (and only if) locally an area exists for which triangles with equal sides have equal area, then the space is a locally isometric map of either the euclidean plane, or a hyperbolic plane, or a sphere.*

Consequently, Hero's and the corresponding non-euclidean formulas[2] are (up to constant factors) the only possible formulas for area in terms of the sides, and each formula is characteristic for its respective geometry.

II. *If (and only if) locally an area* α *exists such that the area of the triangle* pab *depends only on* p, *the local branch of the geodesic* $\mathfrak{g}$ *that contains the segment* $\mathfrak{s}(a, b)$, *and the distance* ab (the euclidean geometry is, of course, the special case $\alpha = p\mathfrak{g}\cdot ab/2$), *then the space is a locally isometric map of a Minkowski plane.*

The exact hypotheses regarding the space R are these: (1) *R is a metric space.* (2) *R is finitely compact.* (3) *R is two-dimensional.* (4) *R is convex.* If xy denotes the distance of x and y, let (xyz) denote the statement that the three points x, y, z are different and that $xy+yz=xz$. (5) *Every point p has a neighborhood $U(p)$ such that for any two different points x, y in $U(p)$ a point z with (xyz) exists.* (6) *If (xyz_1), (xyz_2) and $yz_1 = yz_2$, then $z_1 = z_2$.*[3]

The following facts are known to hold in R: If $S(p, \rho)$ denotes the set of points x with $px < \rho$, then a $\rho(p) > 0$ exists such that: $S(p, 3\rho(p))$ is homeomorphic to a circular disk [1, p. 29]. The segment $\mathfrak{s}(a, b)$

Received by the editors October 10, 1946.

[1] The assumptions are formulated further on. They are equivalent to the conditions A, B, C, D in [1, pp. 11, 12]. Numbers in brackets refer to the references cited at the end of the paper.

[2] In spherical geometry L'Huilier's formula for the defect can be used; for a proof see [4, p. 134]. The analogous hyperbolic formula may be obtained in the same manner. A slightly different form is found in [5, p. 129].

[3] The requirements (5) and (6) are equivalent to D in [1, p. 12] or [2, p. 215], see [2, Theorem (4.1)].