A NOTE ON THE INTERRELATION OF SUBSETS OF INDEPENDENT VARIABLES OF A CONTINUOUS FUNCTION WITH CONTINUOUS FIRST DERIVATIVES[1]

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Let X be a set of n variables $x_1, x_2, \cdots, x_n$ and y a continuous function of these variables, $y=f(x_1, x_2, \cdots, x_n)$, also written as $f(X)$ with continuous partial derivatives of the first, second, and third orders. Throughout the following discussion all the first partial derivatives of $f(x_1, x_2, \cdots, x_n)$ are assumed to be different from zero.

Let S be a proper subset of X and S' the subset which is complementary to S in X.

DEFINITION I. A subset S of independent variables is locally functionally separable at a point $(a_1, a_2, \cdots, a_n)$ within the set X in $y=f(X)$ if there exist some function ϕ with continuous first derivatives and defined in some neighborhood of $(a_1, a_2, \cdots, a_v)$ and another function ψ also with continuous first derivatives and defined in some neighborhood of $(b, a_{v+1}, a_{v+2}, \cdots, a_n)$ where $b=\phi(a_1, a_2, \cdots, a_v)$ such that

$$y = f(X) = \psi(\phi(S), S'). \tag{1}$$

The function $\phi(S)$ is locally separable in $f(X)$.

From the definition of a locally separable subset it follows that each of the original variables, that is, each of the elements of X, is a locally separable subset of X in $f(X)$; so is also the set X itself.

Let ${}_{ij}R$ be defined by

$${}_{ij}R = f_i'/f_j'$$

where f_i' and f_j' are the partial derivatives of $f(x_1, x_2, \cdots, x_n)$ with respect to x_i and x_j. The necessary and sufficient condition for ${}_{ij}R$ to be independent of x_k is

$${}_{ij}R_k' = \frac{f_{ik}''f_j' - f_{jk}''f_i'}{[f_j']^2} = 0$$

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[1] The subject of this note suggested itself by a study of "production functions" and "consumption functions" in mathematical economics.

The problem considered in this paper bears some relation to the XIIIth problem of Hilbert: D. Hilbert, Nach. Ges. Wiss. Göttingen (1900) p. 280; cf. also Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. I, problems 119 and 119a, pp. 61–62 and 220–222.