OPEN TRANSFORMATIONS AND DIMENSION[1]

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This paper considers separable metric spaces A and B and open transformations. If, for each $x \in A$, $f(x) \in B$ and the image under f of every open set in A is a set open in B, then f is an *open* transformation. Continuity of f is not assumed. Such transformations have been studied by Rhoda Manning [1].[2]

THEOREM 1. *If $f(A)=B$ where f is open, then there exists a subset A_1 of A such that* (1) *$f(A_1)=B$,* (2) *for $y \in B$, the set $f^{-1}(y) \cdot A_1$ is countable, and* (3) *f, considered as a transformation of A_1 into B, is open.*

PROOF. Let $K_1, K_2, \cdots$ denote the elements of a countable base (open sets) for the space A. For every $y \in B$ and each i let P_{yi} be a point of $K_i \cdot f^{-1}(y)$, provided this set is nonvacuous. Let A_1 be the set of all points P_{yi} so obtained. Properties (1) and (2) are obvious. To prove (3), let V be an open set in A_1, and U an open set in A such that $U \cdot A_1 = V$. Now for every y the set $f^{-1}(y) \cdot A_1$ is dense in $f^{-1}(y)$. Hence if $f^{-1}(y)$ has a point in U then it has a point in $A_1 \cdot U$ so $f(V)=f(U)$ is an open set in B.

THEOREM 2. *There exist countable-fold open mappings*[3] *which increase dimension.*

PROOF. There exist open mappings which increase dimension [2]. Theorem 2 follows by applying Theorem 1 to any such example.[4]

THEOREM 3. *If* $\dim A = n$ *and* $-1 < m \leqq n$, *then there exists a B and a transformation f such that* (1) $f(A)=B$, (2) *f is open and* 1-1, *and* (3) $\dim B = m$. *In other words, dimension can be lowered at will by a* 1-1 *open transformation.*[5]

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[1] One statement in the abstract (Bull. Amer. Math. Soc. Abstract 59-5-210) is incorrect. Theorem 4 gives the correct statement.

[2] Numbers in brackets refer to the bibliography.

[3] A *mapping* is a continuous transformation.

[4] Alexandroff [5] has proved that if A is *compact* then no countable-fold open mapping can increase dimension.

[5] Compare the following special case of a theorem of Hurewicz [4, p. 91, Theorem VI 7]): "If f is a *closed mapping* of A into B and for each $y \in B$, $f^{-1}(y)$ is zero-dimensional, then $\dim B \geqq \dim A$." In a footnote (loc. cit.) the authors state that it is not known if Theorem VI 7 is true for *open* mappings. The answer is in the negative and their example VI 10 is a counter example, as the mapping f is actually open.