ON THE SINGULARITIES OF A CLASS OF FUNCTIONS ON THE UNIT CIRCLE

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Pólya has suggested and Szegö and others have proved the following theorem.[1]

THEOREM. *Let $f(z)$ be a function regular in the whole plane including $z=\infty$ except at $z=1$. Let*

$$f(z) = \begin{cases} \sum a_n z^n, & |z| < 1, \\ \sum b_n/z^n, & |z| > 1. \end{cases}$$

If $a_n = O(n^k)$ and $b_n = O(n^k)$ then $f(z)$ is a rational function.

The above theorem is generalized in this paper as follows.

THEOREM 1. *Let $f(z)$ be regular in the whole plane including $z=\infty$, except possibly at a certain set S of points on $|z|=1$ (the set S being not everywhere dense on the complete circumference of the unit circle). Let*

$$f(z) = \begin{cases} \sum a_n z^n, & |z| < 1, \\ \sum b_n/z^n, & |z| > 1, \end{cases}$$

and let $a_n = O(n^k)$, $b_n = O(n^k)$; then the following results hold.

(i) *Every isolated singularity on $|z|=1$ will be a pole of order not exceeding $k+1$.*

(ii) *If there are only a finite number of singularities on $|z|=1$, then $f(z)$ is a rational function.*

THEOREM 2. *There exists a function satisfying the hypothesis of Theorem 1 and having an infinite number of singularities on the unit circle; also there exists a function satisfying the same hypothesis and having no isolated singularities.*

LEMMA 1. *Let $f(z)$ be an integral function and let*

$$I_p(r) = \int_0^{2\pi} |f(re^{i\phi})|^p d\phi \qquad \textit{where } p > 0$$

be bounded on a sequence of circles $r=r_n$ tending to infinity, for some $p>0$. Then $f(z)$ reduces to a constant.

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[1] J. Deutschen Math. Verein vol. 40 (1931) Aufgaben und Lösungen p. 81 (Polyá); ibid. vol. 43 (1934) Aufgaben und Lösungen pp. 13–16 (Szegö and others).