ON THE LOWER ORDER OF INTEGRAL FUNCTIONS

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Let $f(z)=\sum_0^\infty a_n z^n$ be an integral function of order ρ. It is known that[1]

$$\text{(1)}\quad \limsup_{n\to\infty} \frac{n\log n}{\log\{1/|a_n|\}} = \rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} \qquad (0 \leqq \rho \leqq \infty).$$

A similar result for the lower[2] order λ, namely

$$\liminf_{n\to\infty} \frac{n\log n}{\log\{1/|a_n|\}} = \lambda = \liminf_{r\to\infty} \frac{\log\log M(r)}{\log r},$$

does not always hold. In fact for

$$\exp(z^2) + \exp(z) = 2 + z + z^2\left(\frac{1}{1!} + \frac{1}{2!}\right) + \cdots,$$

$$\liminf_{n\to\infty} \frac{n\log n}{\log\{1/|a_n|\}} = 1$$

whereas $\lambda=\rho=2$.

We prove here the following theorem.

THEOREM 1. *If $f(z)=\sum_0^\infty a_n z^n$ is an integral function of order ρ and lower order λ $(0\leqq\lambda\leqq\infty)$ then*

$$\text{(2)}\qquad \lambda \geqq \liminf_{n\to\infty} \frac{n\log n}{\log\{1/|a_n|\}} \geqq \liminf_{n\to\infty} \frac{\log n}{\log|a_n/a_{n+1}|}.$$

COROLLARY 1.[3]

$$\text{(3)}\qquad \begin{aligned} &\liminf_{n\to\infty} \frac{\log|a_n/a_{n+1}|}{\log n} \leqq \liminf_{n\to\infty} \frac{\log\{1/|a_n|\}}{n\log n} = \frac{1}{\rho} \leqq \frac{1}{\lambda} \\ &\leqq \limsup_{n\to\infty} \frac{\log\{1/|a_n|\}}{n\log n}; \leqq \limsup_{n\to\infty} \frac{\log|a_n/a_{n+1}|}{\log n}. \end{aligned}$$

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[1] E. C. Titchmarsh, *Theory of functions*, pp. 253–254; E. T. Copson, *Theory of functions of a complex variable*, pp. 175–178.

[2] For the definition, and so on, see (i) J. M. Whittaker, *The lower order of integral functions*, J. London Math. Soc. vol. 8 (1933) pp. 20–27; (ii) S. M. Shah, *The lower order of the zeros of an integral function* (II), Proceedings of the Indian Academy of Sciences (A) vol. 21 (1945) pp. 162–164.

[3] Cf. a similar result (1) in S. M. Shah, *The maximum term of an entire series*, Mathematics Student vol. 10 (1942) pp. 80–82.