NOTE ON ALMOST-ALGEBRAIC NUMBERS

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1. **Introduction.** According to a theorem of J. Liouville,[2] if θ is an algebraic number of degree n, then any approximation by rationals, p/q, is of such a nature that

$$|\theta - p/q| \geqq kq^{-n} \tag{1}$$

for a positive constant k. Liouville constructed his transcendental numbers as the limit of special sequences of rationals, p/q, which violated condition (1) regardless of the values of k and n, as $q \rightarrow \infty$. Thus Liouville constructed *almost-rational* numbers.

E. Maillet[3] likewise found a lower bound for $\theta - \alpha$ where now θ is approximated by the quadratic numbers, α. He then violated his lower bound by substituting for θ the value of an almost periodic simple continued fraction and for α a quadratic number, namely a periodic simple continued fraction that θ almost represented. Thus he constructed an *almost-quadratic* transcendental.

It is an elementary matter to find a lower bound for $\theta - \alpha$, where we now approximate θ by an algebraic number not necessarily rational or quadratic. We could then try several departures. We could, for example, try to construct almost-cubic or almost-biquadratic transcendentals.[4] On the other hand, we could use a *diagonal* method, that is, we could consider the limit of a rapidly converging sequence of algebraic numbers whose degree becomes indefinite. For example, a root of a power series with rational coefficients is the limit of a sequence of (algebraic) roots of the partial sums, and the speed of convergence is regulated by the remainder. If the remainder is too small we find that the root of our power series can be approximated too closely by algebraic numbers of varying degrees, namely the roots of

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[2] J. Liouville, *Sur des classes très étendues des quantités dont la valeur n'est ni algébrique ni même réductible à des irrationelles algébriques*, J. Math. Pures Appl. vol. 16 (1851).

[3] E. Maillet, *Théorie des nombres transcendents et des propriétés arithmétiques des fonctions*, Paris, Gauthier-Villars, 1906, chap. 7.

[4] For instance, E. Maillet, op. cit., pp. 22, 100, considers certain "rapidly converging" power series with rational coefficients and algebraic values of the argument. The value of such a series is shown to be almost algebraic when regarded as the limit of the algebraic partial sums (which lie in the field generated by the argument and therefore are of no higher degree than the argument).