SEMI-SIMPLE EXTENSIONS OF RINGS

OSCAR GOLDMAN

In this paper we investigate the conditions under which a given ring is a sub-ring of a semi-simple ring.[1] For convenience, we say that a ring A is an *extension* of a ring B, if B is a sub-ring of A. It is found that the existence of a semi-simple extension is equivalent to the vanishing of the extension radical, a two-sided ideal defined analogously to the ordinary radical. In Theorem II we give an intrinsic characterization of the extension radical, where we find that the latter is determined by the addition in the ring and is independent of the multiplication. This result is summarized in Theorem III.

For the convenience of the reader, we reproduce here some of the definitions given in the paper mentioned in footnote 1. The *radical* of a ring A is obtained as the intersection of the annihilators of all simple A-modules. When the radical consists only of the zero element of A, we say that the ring is *semi-simple.* The radical as defined here contains the ideal classically known as the radical (the sum of all nilpotent left ideals) and is equal to it if one assumes tht A satisfies the descending chain condition on left ideals. A ring which is semi-simple in the present sense has then no nilpotent ideals.

In order to define the extension radical, we must first introduce the auxiliary notion of a quasi-simple module. If $\mathfrak{M}$ is an abelian group, denote by $E(\mathfrak{M})$ the ring of all endomorphisms of $\mathfrak{M}$. We say that $\mathfrak{M}$ is a *quasi-simple group* if $\mathfrak{M}$ is a simple $E(\mathfrak{M})$-module. An A-module $\mathfrak{M}$ is a *quasi-simple module* if the underlying additive group of $\mathfrak{M}$ is quasi-simple. We shall find the following two lemmas useful.

LEMMA I. *If $\mathfrak{M}$ is a simple A-module, then $\mathfrak{M}$ is a quasi-simple module.*

Let $\mathfrak{A}$ denote the annihilator of $\mathfrak{M}$. It is clear that $\mathfrak{M}$ is a simple $A/\mathfrak{A}$-module, and furthermore that $A/\mathfrak{A} \subset E(\mathfrak{M})$. Thus $\mathfrak{M}$ is certainly simple for $E(\mathfrak{M})$.

LEMMA II. *Every quasi-simple group is the underlying additive group of a vector space over a field and conversely.*

Let Z be the center of $E(\mathfrak{M})$. Since $\mathfrak{M}$ is a simple $E(\mathfrak{M})$-module, Z is

Received by the editors June 6, 1946.

[1] For the definitions and elementary properties of the concepts involved in the study of semi-simple rings, see my previous paper, *A characterization of semi-simple rings*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 1021–1027.