A NOTE ON THE AREA OF A NONPARAMETRIC SURFACE

HARRY D. HUSKEY

1. **Introduction.** Recently, Besicovitch [1][1] has exhibited a surface of the form $z=f(x, y)$ for which sequences of inscribed polyhedra, corresponding to triangulations into nearly isosceles right triangles, do not converge in area to the area of the surface. On the other hand, for surfaces defined by functions $f(x, y)$ which are absolutely continuous in the sense of Tonelli this paper shows (see Theorem I) that in a statistical sense sequences of inscribed polyhedra that arise from triangulations which are successive refinements and which consist of right triangles of any preassigned shape (exactly isosceles, for example) will converge in area to the area of the surface. For surfaces of a more general class (defined by functions which are absolutely continuous in the sense of Young, see below) analogous but somewhat weaker theorems (see Theorems II and III) are proved.

In order to state the results more precisely let us consider the following definitions. Let Q_0 denote the unit square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$. Let $\bar{S}$ denote the continuous surface defined by the continuous function $z=f(x, y)$ for $-\infty < x < +\infty$, $-\infty < y < +\infty$. Suppose that $f(x, y)$ is periodic of period one in x and in y. The results of this paper are valid for a continuous function defined only on Q_0 since by extension of definition and a suitable change of scale (see Saks [8, p. 170]) the above conditions can be satisfied.

Let $D_n^0(u, v)$, for $0 \leqq u \leqq 1$ and $0 \leqq v \leqq 1$, denote a subdivision of Q_0 into rectangles formed by the lines $x=u+i/n$ and $y=v+i/n$ where i takes on all integral values (positive, negative, and zero) which give lines across Q_0. Let (x', y'), (x', y''), (x'', y'), and (x'', y'') denote the vertices of a generic rectangle with $x' < x''$ and $y' < y''$. Let

$$T_1(f; x', y') = \frac{1}{2}\left\{1 + \left[\frac{f(x'', y') - f(x', y')}{x'' - x'}\right]^2 + \left[\frac{f(x', y'') - f(x', y')}{y'' - y'}\right]^2\right\}^{1/2}, \tag{1}$$

$$T_2(f; x', y') = \frac{1}{2}\left\{1 + \left[\frac{f(x'', y'') - f(x', y'')}{x'' - x'}\right]^2 + \left[\frac{f(x'', y'') - f(x'', y')}{y'' - y'}\right]^2\right\}^{1/2}. \tag{2}$$

Received by the editors March 7, 1946.

[1] Numbers in brackets refer to the references cited at the end of the paper.