ON THE NUMBER OF 1-1 DIRECTLY CONFORMAL MAPS WHICH A MULTIPLY-CONNECTED PLANE REGION OF FINITE CONNECTIVITY p (>2) ADMITS ONTO ITSELF

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1. **Introduction.** It is well known[1] that a plane multiply-connected region G of finite connectivity greater than two admits only a finite number of 1-1 directly conformal maps *onto* itself (such maps will be termed henceforth *conformal automorphisms* of G); in fact, if G is of connectivity $p(>2)$, then the number of conformal automorphisms of G can in no case exceed $p(p-1)(p-2)$. The object of the present note is to determine the *best* upper bound, $N(p)$, for the number of conformal automorphisms of G as a *function of the connectivity* p. The basic theorems are:

THEOREM A. *The group of conformal automorphisms of a plane region of finite connectivity $p(>2)$ is isomorphic to one of the finite groups of linear fractional transformations of the extended plane onto itself.*

THEOREM B. *If $p(>2)$ is different from* 4, 6, 8, 12, 20, *then* $N(p)=2p$. *For the exceptional values of p, one has*

$$N(4) = 12, \qquad N(6) = N(8) = 24, \qquad N(12) = N(20) = 60.$$

The proofs of these theorems are based upon the following results:[2]

I. *An arbitrary plane region G of finite connectivity p admits a* 1-1 *directly conformal map onto a canonical plane region G^* whose boundary consists of points and complete circles (either possibly absent), in all p in number, and mutually disjoint.*

If $\mathcal{G}$ and $\mathcal{G}^*$ denote the groups of conformal automorphisms of G and G^* respectively, then $\mathcal{G}$ is isomorphic to $\mathcal{G}^*$. Hence for the purposes of the present problem it suffices to consider the canonical regions and their associated groups of conformal automorphisms.

II. *A conformal automorphism of a canonical region G^* admits an extension in definition throughout the extended complex plane as a linear fractional transformation.*

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[1] Cf. G. Julia, *Leçons sur la représentation conforme des aires multiplement connexes*, Paris, 1934. In particular, see pp. 68–69.

[2] Cf. Hurwitz-Courant, *Funktionentheorie*, Berlin, 1929. See pp. 512–520.