THE HYPERSURFACE CROSS RATIO

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Introduction. In my note on *A 5 curve theorem generalizing the theorem of Carnot*,[1] I introduced the notion of curve cross ratio. This extension of the ordinary cross ratio is the simplest situationally invariant (3.4) case of the generalized or hypersurface cross ratio of $n+1$ pairs of hypersurfaces in n-space which is the subject of the following lines. The generalized cross ratio is at the same time a generalization of the resultant of $n+1$ quantics; the connection between cross ratios and resultants occurred to me when reading a paper of P. Humbert.[2]

The properties of the generalized cross ratio, including extensions of some of those of the ordinary cross ratio, will be developed, together with the similar and interdependent properties of an analogous generalization of the intersection of n hypersurfaces to pairs of hypersurfaces, in §3. This section, much of the contents of which is known, is parallel to §§1 and 2 on the ordinary resultant and intersection. In each section, after the definition and fundamental properties, the influence of a rational transformation of coordinates and of permutation, variation and linear combination of the hypersurfaces is studied.

1. **The resultant.** 1.1. DEFINITION. Let $x=(x_0, \cdots, x_n)$ be a point in (complex, or algebraically closed) projective n-space, $n \geqq 0$, and $a=(a_0, \cdots, a_n)$ a system of $n+1$ quantics of positive degrees $\bar{a}_0, \cdots, \bar{a}_n$ in the variables $x_0, \cdots, x_n$. Then the resultant $[a]$ is an irreducible polynomial in the coefficients of a with $[x_k^{\bar{a}_k}]=1$, such that $[a]=0$ if, and only if, an $x \neq 0$ with $a(x)=0$ (that is, $a_0(x)=0, \cdots, a_n(x)=0$ exists. The resultant is unique since the conditions of irreducibility and $[x_k^{\bar{a}_k}]=1$ distinguish it from its powers and multiples respectively.[3]

1.2. *Degree.* $[a]$ is a quantic of degree $\prod_k \bar{a}$ in the coefficients of a_k. Considering $\bar{a}_k$ as degree of the coefficients, we can write $[a]\cdot = \prod \bar{a}$.

Received by the editors November 2, 1944.

[1] Bull. Amer. Math. Soc. vol. 51 (1945) pp. 972–975.

[2] *Sur l'orientation des systèmes de droites*, Amer. J. Math. vol. 10 (1888) pp. 258–281.

[3] Cf. van der Waerden, *Moderne Algebra* vol. 2, 1931, pp. 18–21. As $|\ |$ is used for the absolute value, $[\ ]$ (already generally used in case of the vector product) is preferable for determinants, resultants and intersections. The product of all degrees shall be denoted by $\prod \bar{a}$, and if $\bar{a}_k$ is omitted, by $\prod_k \bar{a}$ ($=1$ for $n=0$, as all void products including 0 degree in 3.4 and 3.8).