MEASURE PRESERVING HOMEOMORPHISMS AT FIXED POINTS

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In an article of a few years ago [2][1] Kerékjártó obtained interesting results about certain types of transformations which he called similitudes. With a few modifications and extensions his methods can be used to gain information about the structure of measure preserving transformations at fixed points. For simplicity the results are formulated for Euclidean n-space although they could easily be given a much more general setting and in particular the relevant ones apply to any n-dimensional manifold on which there is defined a measure satisfying light restrictions. Actually, as in most topological investigations of measure preserving transformations, the main property needed is that a bounded open set can not be carried into a subset of itself such that the difference of the two sets contains interior points.

It is shown that there are compact continua of assorted sizes which contain the fixed point and which are carried into themselves by the transformation. Such continua might, for example, be solid spheres $x_1^2+ \cdots +x_n^2 \leqq r^2$ as in the case of an orthogonal transformation. On the other hand they might be arcs as in the case of the transformation

$$x_1' = 2x_1, \qquad x_{n-1}' = 2x_{n-1}, \qquad x_n' = 1/2^{n-1}x_n,$$

where continua of the type described are intervals on the x_n-axis which include the origin.

The results also show that there are certain points near the fixed point which remain near it under indefinite positive iteration of the transformation. We use the symbol U^{-1} for the set $T^{-1}(U)$, and so on.

THEOREM 1. *Let T be a measure preserving homeomorphism of E_n onto itself, and let A be a compact connected set such that $T(A) \subset A$. Then if U is an open set with compact closure which includes A, there exists a compact connected set K of which A is a proper subset and such that K is in $\overline{U}^{-1}$ and $T(K) \subset K$.*

The theorem applies to the particular case where A consists of a single fixed point. We divide the proof into two cases.

Case I. Assume that there exists an open connected set V in U^{-1}

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.