ASYMPTOTIC DISTRIBUTION OF SUMS OF RADEMACHER FUNCTIONS

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1. **Introduction.** The central limit theorem of the calculus of probability applied to the special case of the Rademacher functions $r_k(t) = \text{sign} (\sin 2^k\pi t)$ can be stated as follows:

$$\lim_{n\to\infty} \left| \underset{t}{E} \left\{ \frac{1}{(2n)^{1/2}} \left| \sum_{k=1}^{n} r_k(t) \right| < a \right\} \right| = \frac{1}{\pi^{1/2}} \int_{-a}^{a} e^{-u^2} du.$$

This form of the theorem suggests two generalizations, the first of which consists in replacing the constant a by a function $f(t)$. The second generalization is obtained by replacing the constant upper limit of summation n by a sequence of integral-valued functions $N_n(t)$, so that the number of Rademacher functions in the sum varies with t. These are proved by combining the standard techniques of the calculus of probability with the methods of orthogonal series. Both proofs make use of the continuity theorem of Fourier-Stieltjes transforms:

If $f(t) = \lim_{n\to\infty} f_n(t)$, *and if*

$$\lim_{n\to\infty} \int_0^1 \exp\,[ixf_n(t)]dt = e^{-x^2/4}$$

uniformly in x, *then*

$$\lim_{n\to\infty} \left| \underset{t}{E} \left\{ \left| f_n(t) \right| < a \right\} \right| = \frac{1}{\pi^{1/2}} \int_{-a}^{a} e^{-u^2} du.$$

Essential use is made of the Walsh-Kaczmarz system[1] of orthonormal functions, which have two useful properties: (1) each of them is equal to a finite product of Rademacher functions, and (2) any Lebesgue-square integrable function can be expanded in a series of these functions, which series will converge in the mean with index two.

2. **First generalization.** The first theorem expresses in asymptotic form the measure of the point set over which the sum of the Rademacher functions is bounded by a given function.

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[1] Cf. S. Kaczmarz and H. Steinhaus, *Le système orthogonal de M. Rademacher*, Studia Mathematica vol. 2 (1930) and J. L. Walsh, *A closed set of normal orthogonal functions*, Amer. J. Math. vol. 45 (1923) pp. 5–24.