ON THREE PROBLEMS CONCERNING NIL-RINGS

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1. Introduction. In the present note three problems concerning nil-rings are proposed and certain relations linking these problems to one another are discussed.

First problem. The sum of all two-sided[1] nil-ideals of a ring S has been defined by G. Koethe [**2**, §3][2] as the radical of S, provided that this sum contains also all one-sided nil-ideals of S. We shall henceforth refer to this radical as the K-radical[3] of S. It is an open question whether or not there are rings in which the K-radical does not exist.

Second problem. A ring T is called semi-nilpotent (see [**3**, §2]) if each finite set of elements in T generates a nilpotent ring. A ring which is not semi-nilpotent is called semi-regular. Each semi-nilpotent ring is evidently a nil-ring. It is an open question whether or not there exist semi-regular nil-rings. As may easily be seen, this problem is equivalent to the question whether or not there exist semi-regular nil-rings which are generated by a finite set of elements.

Third problem. A nil-ideal P of a ring S has been termed by R. Baer [**1**, §1] a radical ideal if the quotient-ring S/P does not contain nilpotent ideals other than zero. The sum $U(S)$ and the crosscut $L(S)$ of all radical ideals of a ring S are again radical ideals which are called the upper radical and the lower radical respectively (see Baer [**1**, §1]). As indicated by Baer ideals may exist between $U(S)$ and $L(S)$ which are not radical ideals. R. Baer has also constructed an interesting example which illustrates this possibility. Our results in the present note show that this phenomenon can not be considered as an exception to the rule but on the contrary rather as the rule itself, and thus the following problem presents itself: Are there or are there not rings S in which $U(S) \supset L(S)$ and in which furthermore each ideal which lies between $U(S)$ and $L(S)$ is also a radical ideal?

In the present note the following results are obtained: Suppose that S is a ring in which the K-radical does not exist, then S contains an infinite number of right ideals as well as of left ideals which are semi-regular nil-rings (see Theorem 4 in §3). Suppose that S is a ring with a semi-regular upper radical $U(S)$, then S contains a subring S' so that $U(S) \supseteq S' = U(S') \supset L(S')$ and so that S' contains an infinite

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[1] We shall write henceforth in short *ideals* instead of *two-sided ideals*.

[2] Numbers in brackets refer to the Bibliography at the end of the paper.

[3] For the sake of convenience we shall reserve in this paper the term radical for the sum of all semi-nilpotent ideals of the ring (see [**3**, §2]).