SOME REMARKS ON EULER'S ϕ FUNCTION AND SOME RELATED PROBLEMS

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The function $\phi(n)$ is defined to be the number of integers relatively prime to n, and $\phi(n)=n\cdot\prod_{p|n}(1-p^{-1})$.

In a previous paper[1] I proved the following results:

(1) The number of integers $m\leqq n$ for which $\phi(x)=m$ has a solution is $o(n[\log n]^{\epsilon-1})$ for every $\epsilon>0$.

(2) There exist infinitely many integers $m\leqq n$ such that the equation $\phi(x)=m$ has more than m^c solutions for some $c>0$.

In the present note we are going to prove that the number of integers $m\leqq n$ for which $\phi(x)=m$ has a solution is greater than $cn(\log n)^{-1}\log\log n$.

By the same method we could prove that the number of integers $m\leqq n$ for which $\phi(x)=m$ has a solution is greater than $n(\log n)^{-1}(\log\log n)^k$ for every k. The proof of the sharper result follows the same lines, but is much more complicated. If we denote by $f(n)$ the number of integers $m\leqq n$ for which $\phi(x)=m$ has a solution we have the inequalities

$$n(\log n)^{-1}(\log\log n)^k < f(n) < n(\log n)^{\epsilon-1}.$$

By more complicated arguments the upper and lower limits could be improved, but to determine the exact order of $f(n)$ seems difficult.

Also Turán and I proved some time ago that the number of integers $m\leqq n$ for which $\phi(m)\leqq n$ is $cn+o(n)$. We shall give this proof, and also discuss some related questions:

LEMMA 1. *Let $a<\epsilon$, $b<n$, $a\neq b$, $\epsilon=(\log\log n)^{-100}$. Then the number of solutions $N_n(a, b)$ of*

$$(1)\qquad (p-1)a=(q-1)b,\qquad p\leqq na^{-1},\qquad q\leqq nb^{-1},$$

p, q primes, does not exceed

$$(2)\qquad \frac{(a, b)}{ab}\frac{n}{(\log n)^2}(\log\log n)^{80}.$$

PROOF. Put $(a, b)=d$. Then we have $p\equiv 1 \bmod bd^{-1}$. Also $(p-1)ab^{-1}+1=q$ is a prime. We can assume that both p and q in (1) are greater

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[1] *On the normal number of prime factors of $p-1$*, Quart. J. Math. Oxford Ser. vol. 6 (1935) pp. 205–213.