THE BASIS THEOREM FOR VECTOR SPACES OVER RINGS

C. J. EVERETT

It is the purpose of this note to establish the following theorem:

THEOREM. *A vector space* $M = u_1K + \cdots + u_mK$ *of* m *basis elements over a ring* $K = \{0, a, b, \cdots, 1\}$ *with unit* 1 *has the property that every subspace* $N > 0$ *possesses a basis of* $n \leqq m$ *elements if and only if* K *is a right principal-ideal-ring without zero-divisors.*

That such a ring insures the basis condition for subspaces is well known [3, p. 121].[1]

Suppose now that every subspace $N > 0$ has a basis of $n \leqq m$ elements. It has been shown [2, Theorem (F)] that every right ideal $R > 0$ of K must then have a single generator: $R = r_0K$, where $r_0k = 0$ implies $k = 0$. Moreover, since every right ideal has a finite set of generators, the ascending chain condition must hold for right ideals of K [3, p. 26]. It therefore suffices to prove the following two lemmas.

LEMMA 1. *In a ring* K *with unit* 1 *and ascending chain condition for right ideals, equations* $ab = 1$, $ac = 0$ *imply* $c = 0$.

If $c \neq 0$, the linear transformation $k \rightarrow ak$, $k \in K$, would be of type (iv) [2, p. 313], that is, $K/K_0 \cong K$, and $0 < K_0 < K_1 < K_2 < \cdots$, where K_i is defined inductively as the set of all elements of K mapped into elements of K_{i-1}. This contradicts the chain condition.

LEMMA 2. *A ring* K *with unit in which every right ideal* $R > 0$ *is of the form* r_0K, *where* $r_0k = 0$ *implies* $k = 0$, *has no zero divisors.*

Let $sc = 0$, $s \neq 0$, and $sK = r_0K \neq 0$, where $r_0k = 0$ implies $k = 0$. We have $s = r_0a$, $r_0 = sb = r_0 \cdot ab$, $r_0(ab - 1) = 0$, and hence $ab = 1$. Also, $sc = 0 = r_0ac$, and $ac = 0$. Since Lemma 1 applies to K, $c = 0$.

It should be noted that the result follows also from a result of Baer's [1, Theorem 5 or Lemma 4] which states that in a ring with unit and weak maximal condition, $ab = 1$ implies $ba = 1$.

BIBLIOGRAPHY

1. R. Baer, *Inverses and zero-divisors*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 630–638.

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[1] Numbers in brackets refer to the bibliography.