A REMARK ON METRIC BOOLEAN RINGS

MALCOLM F. SMILEY

The purpose of this note is to prove that if, on a ring $B \equiv [a, b, c, \cdots]$ with unity element 1, a real valued function $\mu(a)$ is defined satisfying

$$\mu(a) > 0 \quad \textit{for every} \quad a \neq 0, \tag{1}$$

$$\mu(a + b) + 2\mu(ab) = \mu(a) + \mu(b) \tag{2}$$

for every $a, b \in B$, then B is a metric Boolean ring[1] [**2**, pp. 41 and 96]. This result is analogous to one of Glivenko's [**3**] which states that every metric lattice is modular [**2**, p. 42]. We discuss also the following modification of (1):

$$\mu(a) \geqq 0 \quad \textit{for every} \quad a \in B. \tag{3}$$

The conditions (2) and (3) also lead, via identification, to a metric Boolean ring.

THEOREM 1. *Let B be a ring with unity element* 1 *on which is defined a real valued function $\mu(a)$ satisfying* (1) *and* (2). *Then B is a metric Boolean ring.*

The following lemma lists the steps in our proof of Theorem 1.

LEMMA 1. *For every $a, b \in B$, we have* (i) $\mu(a) = 0$ *if and only if* $a = 0$, (ii) $\mu(ab) = \mu(ba)$, (iii) $\mu(1+a) = \mu(1) - \mu(a)$, (iv) $\mu(a^2b) = \mu(ab^2)$, (v) $\mu(a^2) = \mu(a)$, (vi) $a + a = 0$, (vii) $a^2 = a$.

PROOF. (i) Set $b = 0$ in (2) and use (1). (ii) This is clear by (2). (iii) Set $b = 1$ in (2). (iv) From (2) and (iii) we have

$$\mu(a + b + 1) + 2\mu(ab + a) = \mu(a) + \mu(1) - \mu(b).$$

Using (2) again gives

$$\mu(a + b + 1) + 2\mu(ab) + 2\mu(a) - 4\mu(a^2b) = \mu(a) + \mu(1) - \mu(b).$$

Rearranging, we find that

$$4\mu(a^2b) = \mu(a + b + 1) + 2\mu(ab) + \mu(a) + \mu(b) - \mu(1).$$

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[1] Numbers enclosed in brackets denote references given at the end of the paper.