LINEAR TOPOLOGICAL SPACES

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1. Introduction. In general analysis it is customary to study linear spaces for which there is defined a "norm," which takes the place of the absolute value of ordinary analysis in defining distance, limit point, continuity, and so on. Linear metric spaces more general than the normed spaces have also been studied by Fréchet, Banach, and several others. In line with the trend toward general topology, it seems natural to generalize still more by introducing *linear topological spaces*, that is, linear spaces which are at the same time topological spaces, in which the fundamental "linear" operations of addition and scalar multiplication are continuous. We shall always assume that the topology is subject to the axioms for a T_1-space.[1]

The topology of T_1-spaces may be introduced in various ways; by postulating a system of open sets or of neighborhoods with certain properties, and so on. We shall find it convenient to give a set of postulates for the topology of the linear space L in which "neighborhood" is the fundamental undefined notion. Since L is a topological group, it has a uniform topology and hence it is sufficient to consider neighborhoods of the origin. Moreover, the "uniform structure" implies that L is a completely regular Hausdorff space.[2]

The following notations will be used. The set of elements x having the property P will be denoted by $\{x; P\}$. If S and T are subsets of L, α is a fixed real number, and x a fixed point of L, $x+S$ denotes the set $\{x+y; y\in S\}$; $S+T$ denotes the set $\{y+z; y\in S, z\in T\}$; αS stands for the set $\{\alpha y; y\in S\}$. The notations $\cup$ and $\cap$ are used for union and intersection, respectively. The following definition of a linear topological space is equivalent to the one given above.[3]

DEFINITION 1.1. A linear space L will be called a *linear topological space* (abbreviated l.t.s.) if and only if there is a system $\mathcal{V}$ of subsets

An address delivered before the April meeting of the Society in Berkeley on April 29, 1944, by invitation of the Program Committee; received by the editors July 17, 1944.

[1] The definition given here seems to be due to Kolmogoroff [1], 1934. However, a linear space with a more general topology than that of a T_1-space was defined by Fréchet [2, 3] in 1926 under the name "topological affine space." The postulates for a T_1-space are given in Alexandroff and Hopf [1, p. 59]. The bracketed numbers refer to the bibliography.

[2] A. Weil [1, p. 13].

[3] For the proof see Hyers [4].