AN ELEMENTARY PROOF OF THE STRONG FORM OF THE CAUCHY THEOREM

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A difficult step in the derivation of the strong forms of the Cauchy theorem, Green's lemma, and related theorems from the corresponding weak forms is the construction, for a given rectifiable Jordan curve J, of a sequence of Jordan polygons lying interior to J, converging to J, and having uniformly bounded lengths. This note presents what the author believes to be a simpler elementary construction of this sequence than any hitherto available (see [1, 2][1] and the bibliographies at the ends of these papers). We shall illustrate its use by proving the strong Cauchy theorem.

We assume the Jordan separation theorem together with its elementary consequences, as is usual in proofs of this kind, and we assume the weak form of the Cauchy theorem ($\int_P f(z)dz=0$ if $f(z)$ is analytic in a region containing P and its interior) for Jordan polygons P whose edges lie on lines of the form $x=m2^{-N}$, $y=n2^{-N}$, where m, $n=0, \pm 1, \cdots$. The polygon P and its interior is of course the sum of a finite number of squares of the network of closed squares (with sides 2^{-N}) into which these lines cut up the plane, so that we need to assume the weak Cauchy theorem only for single squares.

Let X be a fixed interior point of the given rectifiable Jordan curve J with coordinates *not* of the form $m2^{-N}$, and let C be a fixed closed square lying interior to J, containing X and having sides parallel to the axes. Let I be the set of squares of the above network lying interior to J; for sufficiently large N every square of I which contains a point of C lies interior to J. The vertical ray from X cuts a first edge l_1 of a square not in I. We form a polygon P by proceeding to the left along l_1 from its right-hand end point p_0, and at any vertex p_n choosing as l_{n+1} from the three remaining edges the counter-clockwise-most one which has a square of I on its left and a square not in I (and so containing a point of J) on its right. We shall call the latter square S_{n+1}. The reader will find the following argument trivial if he will sketch the four possible configurations at p_n. In tracing P a first vertex p_n must be repeated, $p_n=p_{n+M}$. If $n>0$, then l_n and l_{n+M} approach p_n from opposite directions, $S_{n+1}=S_{n+M}$, and the Jordan polygon l_{n+1} to l_{n+M} separates S_{n+1} from S_n and so separates points of J, an impossibility. Thus $n=0$, $p_M=p_0$ and P is a Jordan polygon. Note

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[1] Numbers in brackets refer to the bibliography at the end of the paper.