THE "FUNDAMENTAL THEOREM OF ALGEBRA" FOR QUATERNIONS

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We are concerned with polynomials of degree n of the type

$$f(x) = a_0 x a_1 x \cdots x a_n + \phi(x),$$

where $x, a_0, a_1, \cdots, a_n$ are real quaternions ($a_i \neq 0$ for $i = 0, 1, \cdots, n$) and $\phi(x)$ is a sum of a finite number of similar monomials $b_0 x b_1 x \cdots x b_k$, where $k < n$.

THEOREM 1. *The equation $f(x) = 0$ has at least one solution.*[1]

The 4-dimensional euclidean space R_4 of all quaternions will be made compact by adding the point ∞ to form a 4-dimensional sphere S_4. Setting $f(\infty) = \infty$ we get a mapping

$$f: S_4 \to S_4.$$

The continuity of f at ∞ follows from the fact that $|f(x)|$ increases without limit as $|x|$ increases without limit, a fact which is obvious from the definition of f. It should be noted that this argument is not valid for polynomials of degree n with more than one term of degree n.

Theorem 1 is then a consequence of the following theorem which asserts that "essentially" the equation $f(x) = 0$ has exactly n solutions.

THEOREM 2. *The mapping $f: S_4 \to S_4$ has the degree n (in the sense of Brouwer).*[2]

We define a mapping $g: S_4 \to S_4$ as follows:

$$g(x) = x^n \quad \text{for} \quad x \in R_4, \qquad g(\infty) = \infty.$$

Theorem 2 is a consequence of the following two lemmas.

LEMMA 1. *The mappings f and g of S_4 into S_4 are homotopic.*

LEMMA 2. *The mapping g has degree n.*

PROOF OF LEMMA 1. Define for $0 \leqq t \leqq 1$

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[1] This result has been obtained for the special case in which all terms of $f(x)$ have the form ax^k; cf. Ivan Niven, *Equations in quaternions*, Amer. Math. Monthly vol. 48 (1941) pp. 654–661.

[2] Cf. Alexandroff and Hopf, *Topologie*, Berlin, 1935, chap. 12, for the theory of the degree of a mapping.