THE HIGHER COMMUTATOR SUBGROUPS OF A GROUP

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It is not the object of this address to introduce you to new theories or to tell of great discoveries. Quite on the contrary; I intend to speak of unsolved problems and of conjectures. In order to describe these, certain concepts will have to be discussed; and for obtaining a proper perspective it will be necessary to mention a number of theorems, some of them new. The proofs of the latter will be relegated to appendices so that the hurried reader may skip them easily. The bibliography is in no sense supposed to be complete. We just selected convenient references for facts mentioned and beyond that just enough to be a basis for further reading.

1. **The hierarchy of invariant subgroups.** The subgroups of a group may be classified according to the operations which leave them invariant. There are first the *normal* subgroups of the group G, characterized by the fact that they are transformed into themselves by the inner automorphisms of G; and for this reason they had at one time appropriated the term "invariant subgroup." There are next the *characteristic* subgroups of G which are left invariant by every automorphism of G. Clearly not every subgroup of G is normal, unless G belongs to a comparatively special class of groups, the so-called abelian and hamiltonian groups; and neither is in general every normal subgroup characteristic, though this may happen too (for example, in cyclic groups and in simple groups).

These two classes of subgroups are well known, but for our purposes they are too big. There is next the class of subgroups which we shall term for lack of a better name *strictly characteristic.* A subgroup S of G belongs to this class if $S^f \leqq S$ whenever f is an endomorphism[1] of G and $G^f = G$. The distinction between characteristic and strictly characteristic subgroups does not cut very deep, since there exists a very big class of groups with the following property:

(Q) *If f is an endomorphism of G such that $G^f = G$, then f is an automorphism of G.*

This postulate (Q) is satisfied by every finite group and more generally by every group satisfying the ascending chain condition for normal subgroups. But the Q-groups are not exhausted by the groups

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[1] An endomorphism of the group G is a single-valued and multiplicative G to G function.