CAUCHY-STIELTJES AND RIEMANN-STIELTJES INTEGRALS

G. BALEY PRICE

1. Introduction. This note treats the equivalence of the Riemann-Stieltjes and Cauchy-Stieltjes integrals (abbreviated RS and CS integrals) and conditions for the existence and equality of the latter. The ordinary Riemann, the left Cauchy, and the right Cauchy integrals are defined as limits of the sums $\sum_1^n f(\xi_i)(x_i - x_{i-1})$, $x_{i-1} \leqq \xi_i \leqq x_i$, $\sum_1^n f(x_{i-1})(x_i - x_{i-1})$ and $\sum_1^n f(x_i)(x_i - x_{i-1})$ respectively; it is known [2] that these integrals are equivalent. Corresponding to these integrals, we have the RS and the two CS integrals, defined as limits of the sums $\sum_1^n f(\xi_i)[g(x_i) - g(x_{i-1})]$, $\sum_1^n f(x_{i-1})[g(x_i) - g(x_{i-1})]$, and $\sum_1^n f(x_i)[g(x_i) - g(x_{i-1})]$. The right modified RS integral is obtained from the sums $\sum_1^n f(\xi_i)[g(x_i) - g(x_{i-1})]$, $x_{i-1} \leqq \xi_i < x_i$. Examples in §4 show that the CS integrals may exist, with equal or unequal values, when the RS and the right modified RS do not; that the right modified RS integral may exist when the RS does not; and that one of the CS integrals may exist when the other does not. Thus the RS, the right modified RS, and the two CS integrals are not equivalent. Since all these integrals obviously exist when the RS integral does, it is natural to investigate conditions under which the existence of a CS or right modified RS integral implies the existence of the RS integral. It is shown in this note that if g is non-decreasing, if f and g have no common discontinuities on the same side, and if the left CS integral exists, then the RS integral exists and has the same value, the integrals being limits in the sense of increasing refinement of subdivisions. This result is established in two steps: (a) if g is non-decreasing, if f and g have no common discontinuities on the right, and if the left CS integral exists, then the right modified RS integral exists; (b) if the right modified RS integral exists, if g is non-decreasing, and if f and g have no common discontinuities on the left, then the RS integral exists. This result obviously includes the previously proved equivalence of the Riemann and Cauchy integrals [2] and certain others [3]. Further, it states sufficient conditions for the equality of the two CS integrals; these conditions show that the two ordinary Cauchy integrals are always equal. The note closes with a proof that the CS integrals exist when f has only simple discontinuities and g has bounded variation. We conclude from this result and others stated above that both of the CS integrals properly include the RS integral, and that neither CS integral includes the other. Precise statements of

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