CHAINS IN PARTIALLY ORDERED SETS

OYSTEIN ORE

1. **Introduction.** Dedekind [1] in his remarkable paper on *Dualgruppen* was the first to analyze the axiomatic basis for the theorem of Jordan-Hölder in groups. Recently there have appeared a considerable number of investigations in this field; one should mention the contributions by Birkhoff [2], Klein-Barmen [3, 4], Uzkow [5], George [6], Kurosch [7], and the author [8, 9]. Most of these papers deal particularly with the conditions which it is necessary to impose upon a structure or lattice in order to assert that the chains between two elements have the same length.

In the present paper the general problem of comparing chains in partially ordered sets is considered. It seems remarkable and somewhat surprising that one can formulate a theorem about chains in arbitrary partially ordered sets which contains the theorem of Jordan-Hölder as a special case when applied to more restricted systems. This theorem yields for partially ordered sets, hence by specialization also for structures, a necessary and sufficient condition for the chains to have equal lengths.

2. **Simple cycles.** Let us indicate briefly the terminology which we shall use. The basic partially ordered set to be investigated shall be denoted by P. An element x in P *lies between* two elements $a \supset b$ when $a \supset x \supset b$ and *properly between* them when the two possibilities $a = x$ and $b = x$ are excluded. An element a is *prime over* b and b is *prime under* a when $a \supset b$ and there are no elements properly between a and b. A *chain* is an *ordered subset* of P. A *finite chain* between a and b has the form

$$a = a_0 \supset a_1 \supset \cdots \supset a_{n-1} \supset a_n = b$$

where n is the *length* of the chain. A chain is *complete* when it is not possible to intercalate further terms in it. Through transfinite induction one can prove that any chain is contained in some complete chain.

Now let C_1 and C_2 be two complete chains joining two elements $a \supset b$ in such a manner that they have no elements in common except the end points a and b. Such two chains shall be said to form a *simple cycle* when the following condition is fulfilled:

Simple cycle. There shall exist no elements properly between a and b

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