A NOTE ON APPROXIMATION BY RATIONAL FUNCTIONS

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The theory of the approximation by rational functions on point sets E of the z-plane ($z=x+iy$) has been summarized by J. L. Walsh[1] who himself has proved a great number of important theorems some of which are fundamental. The results concern both the case when E is bounded and when E extends to infinity.

In the present note a L_p-theory ($0<p<\infty$) will be given for the following point sets extending to infinity:

A. The real axis $-\infty<x<\infty$, $y=0$.

B. The half-plane $-\infty<x<\infty$, $0<y<\infty$.

The only poles of the approximating functions are to lie at preassigned points whose number will be required to be as small as possible.[2] We shall make use of the theory of the class $\mathfrak{H}_p$ the fundamental results of which are due to E. Hille and J. D. Tamarkin;[3] $\mathfrak{H}_p$ is the set of functions $F(z)$ which, for $0<y<\infty$, are regular and satisfy the inequality

$$\int_{-\infty}^{\infty} |F(x+iy)|^p dx \leqq M^p \quad \text{or} \quad |F(z)| \leqq M$$

for $0<p<\infty$ or $p=\infty$, respectively, where M depends on F and p only. By $|f(x+iy)|_p$ we denote

$$\left(\int_{-\infty}^{\infty} |f(x+iy)|^p dx\right)^{1/p} \quad \text{or} \quad \operatorname*{ess.\,u.b.}_{-\infty<x<\infty} |f(x+iy)|$$

for $0<p<\infty$ or $p=\infty$, respectively, and by α and β two arbitrarily fixed points in the upper or lower half-plane, respectively. We obtain the following results:[4]

THEOREM 1. *Let $0<p<\infty$ and $F(t)\in L_p(-\infty, \infty)$, let c be an integer greater than p^{-1} and $r_k(z)=(\alpha-z)^k(z-\beta)^{-c-k}$ $[k=0, \pm 1, \pm 2, \cdots]$.*

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[1] *Interpolation and approximation by rational functions in the complex domain*, Amer. Math. Soc. Colloquium Publications, vol. 20, 1935.

[2] Compare Walsh, loc. cit., for example, approximation by polynomials.

[3] Fund. Math. vol. 25 (1935) pp. 329–352, $1\leqq p<\infty$. For $0<p<1$ see T. Kawata, Jap. J. Math. vol. 13 (1936) pp. 421–430.

[4] The case $p=\infty$ of each of the results is a special case of Theorem 16, J. L. Walsh, chap. 2.