ON THE FOURIER DEVELOPMENTS OF A CERTAIN CLASS OF THETA QUOTIENTS

M. A. BASOCO

1. **Introduction.** In this paper we shall be concerned with the functions $\phi_\alpha^k(z)$ defined by the relation

$$(1) \qquad \phi_\alpha^k(z) \equiv \left\{\frac{d}{dz}\log \vartheta_\alpha(z, q)\right\}^k = \left\{\frac{\vartheta_\alpha'(z, q)}{\vartheta_\alpha(z, q)}\right\}^k, \qquad \alpha = 0, 1, 2, 3,$$

where $\vartheta_\alpha(z, q)$ is a Jacobi theta function and k is a positive integer. In the first place, we shall derive the Fourier developments which represent these functions in a certain strip of the complex plane; it will be seen that the Fourier coefficients of $\phi_\alpha^k(z)$ depend on those of $\phi_\alpha^s(z)$, $s=1, 2, 3, \cdots, k-1$, through a recurrence relation of order k. Secondly, these developments, in conjunction with certain obvious identities, yield, through the method of paraphrase, some general arithmetical formulae of a type first given by Liouville.[1] Indeed, we recover, in a simple manner, some results given without proof by Liouville, which were later proved by Bell[2] through the use of somewhat complex identities involving a certain set of doubly periodic functions of the second kind. One of these results has recently been proved in a strictly elementary, but very ingenious way, by Uspensky.[3] Finally, we indicate some applications of these formulae to the derivation of a certain type of arithmetic and algebraic identities.

2. **The functions $\phi_\alpha^k(z)$.** It should be pointed out that the case $k=1$, is implicit in §§47 and 48 of Jacobi's *Fundamenta nova*.[4] Likewise, the case $k=2$, has been obtained by G. D. Nichols[5] through the use of certain results due to the present writer.[6] The following is a direct derivation of the necessary procedure for the general case; it depends on a straightforward application of contour integration and the theory of residues. It is convenient to treat the two functions $\phi_0^k(z)$ and

Presented to the Society, November 28, 1942; received by the editors June 29, 1942.

[1] J. Math. Pures Appl. (2) vol. 3 (1858) et seq. See, for example, vol. 3 p. 247 (H).

[2] Trans. Amer. Math. Soc. vol. 22 (1921) p. 215 formula (xiv′).

[3] J. V. Uspensky and M. A. Heaslet, *Elementary number theory*, New York, 1939. See chap. 13 p. 462 formula (R).

[4] Jacobi, *Gesammelte Werke*, vol. 1 p. 187.

[5] G. D. Nichols, Tôhoku Math. J. vol. 40 (1935) pp. 252–258.

[6] M. A. Basoco, Bull. Amer. Math. Soc. vol. 38 (1932) pp. 560–568.