A COMPARISON OF ALGEBRAIC, METRIC, AND LATTICE BETWEENNESS

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Introduction. We propose to investigate here the consequences of the identity of each pair chosen from three important generalizations of the relation of betweenness on a line, namely, algebraic betweenness [**1**, p. 27],[1] metric betweenness [**3**, p. 36], and lattice betweenness [**7**, Part II]. We shall also find an interpretation of metric betweenness in the Banach space of all continuous functions defined on the interval $0 \leqq t \leqq 1$ which can be used to establish the fact that this relation satisfies no strong four or five point transitivity [**7**, Part I] except t_1 and t_2.

We note first that algebraic betweenness implies metric betweenness and lattice betweenness. We find that algebraic betweenness and metric betweenness coincide in a seminormed real vector space[2] if and only if it is strictly convex in the sense of Clarkson [**4**, p. 404]. We then show that the coincidence of metric and lattice betweenness in a semimetric space [**3**, p. 38] which is also a lattice [**2**, p. 16] leads to a system which is a metric lattice (in the sense of G. Birkhoff [**2**, p. 41]). It follows that a complete seminormed real vector lattice is equivalent to an (L)-space [**6**] if and only if its metric and lattice betweenness relations are identical. Finally, we prove that algebraic and lattice betweenness coincide in a real vector lattice if and only if it is equivalent to the system of all real numbers. We conclude by giving the interpretation of metric betweenness in the space[3] $C[0, 1]$.

Presented to the Society, September 10, 1942; received by the editors March 27, 1942.

[1] References to the bibliography at the end of the paper will be in brackets.

[2] We shall use these terms as follows. A *seminormed real vector space* is a vector space over the field of all real numbers together with a real non-negative single-valued function $\|a\|$, called the "norm of a," satisfying (i) $\|\lambda a\| = |\lambda| \|a\|$, and (ii) $\|a\| = 0$ *if and only if* $a = 0$. A *normed real vector space* satisfies in addition (iii) $\|a\| + \|b\| \geqq \|a+b\|$. A *real vector lattice* is a vector space over the field of all real numbers which is also a lattice [**2**, p. 16] with respect to a partial ordering relation "$\geqq$" such that (i) $a \geqq b$ *and* $\lambda \geqq 0$ *implies* $\lambda a \geqq \lambda b$, and (ii) $a \geqq b$ *implies* $a + c \geqq b + c$ *for every* c. A *(semi)normed real vector lattice* is a real vector lattice which is also a (semi)normed real vector space; it is *complete* if evey fundamental sequence has a limit. A complete normed real vector space is usually called a (real) *Banach space.*

[3] The notation $C[0, 1]$ (sometimes simply C) is currently used to designate the space described in the concluding sentence of the preceding paragraph.