ON THE INTERIORITY OF REAL FUNCTIONS[1]

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If X and Y are metric spaces, a continuous transformation $f(X)=Y$ is said to be interior at a point x of X provided that if U is any open subset of X containing x, $f(x)$ is interior to $f(U)$ in Y. In 1928 Kuratowski[2] proved a theorem essentially to the effect that if X is compact, the set G of all points y in Y such that f is interior at all points of $f^{-1}(y)$ is a G_δ-set dense in Y. We first establish the following related result.

THEOREM. *If $f(x)$ is a real-valued continuous function defined on a locally connected separable metric space M (therefore transforming M into a set Y of real numbers), there exists a countable subset C of Y such that f is interior at every point of $M-f^{-1}(C)$.*

PROOF. For each $y \varepsilon Y$ let Y_1 and Y_2 be the sets of all numbers in Y which are less than y and greater than y, respectively. Let $M_1(y)=f^{-1}(Y_1)$, $M_2(y)=f^{-1}(Y_2)$. Now there must exist[3] a countable subset C of Y such that if y is any point of $Y-C$,

$$f^{-1}(y) \subset \overline{M_1(y)} \cdot \overline{M_2(y)}.$$

For if not, there would exist an uncountable subset K of Y such that for any $y \varepsilon K$ there is a point $p_y \varepsilon f^{-1}(y)$ which fails to be a limit point either of $M_1(y)$ or of $M_2(y)$. Clearly we may suppose that for an uncountable subset of K_1 of K, $p_y \cdot \overline{M_1(y)}=0$. Now since M is separable and metric, there exists a countable sequence $R_1, R_2, \cdots$ of open sets in M such that if U is any open set in M and $p \varepsilon U$, there is an m such that $p \varepsilon R_m \subset U$. Hence, for each $y \varepsilon K_1$ there exists an integer m_y such that $p_y \subset R_{m_y}$ and $R_{m_y} \cdot M_1(y)=0$. But if y, $y' \varepsilon K_1$, $y'<y$, we have $p_{y'} \subset M_1(y)$ so that $m_y \neq m_{y'}$. This is absurd, since K_1 was uncountable. Hence we have established the existence of the set C as asserted above.

We shall show that f is interior at any point x of $M-f^{-1}(C)$. To this end let U be any open set in M containing x and let $y=f(x)$. Since M is locally connected, there exists a connected open subset V

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[1] This paper, in slightly different form, was sent to the editors of Fundamenta Mathematicae in June, 1939.

[2] See Fundamentae Mathematicae, vol. 11 (1926), p. 176.

[3] This could be established also with the aid of a lemma of Zarankiewicz. See Fundamenta Mathematicae, vol. 12 (1928), p. 119.