ON THE NILPOTENCY OF THE RADICAL OF A RING

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1. **Introduction.** A few years ago, it was shown by C. Hopkins[2] that the structure theory of noncommutative rings[3] can be based on the assumption of only the minimum condition for left-ideals. Before Hopkins, a maximum condition for ideals had also been used in order to prove that the radical of the ring is nilpotent. Actually this last fact is a special case of the maximum condition, for example, the existence of a maximal nilpotent (two-sided) ideal, and this makes Hopkins' result appear rather surprising.

In this note, I give a short and simple proof for Hopkins' theorem. I also show that it is sufficient to assume only the minimum condition for sets of two-sided nil-ideals (that is, ideals consisting only of nilpotent elements) in order to prove the nilpotency of the radical. The later sections are concerned with the existence of idempotents and primitive left-ideals contained in a given regular left-ideal. Here the assumptions concerning the ring R are those on which Köthe[4] and Deuring[5] based their treatment of noncommutative rings. As was shown by Köthe, these assumptions are equivalent to the validity of the structure theory, so that it is natural to work with them. Once the results of the later sections have been established, there is no difficulty in developing the theory with the usual methods.[6]

2. **Preliminaries.** A ring R is a set of elements for which an addition and a multiplication are defined such that the elements form an abelian group under addition and that the associative law of multiplication and both distributive laws hold. We may also have a set K of operators. Then the product $t\alpha = \alpha t$ of any α in R with any t in K must be defined as an element of R, and the following rules are to hold (α, β in R, t in K)

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[1] Guggenheim Fellow.

[2] Charles Hopkins, Duke Mathematical Journal, vol. 4 (1938), p. 664; cf. also J. Levitzki, Compositio Mathematica, vol. 7 (1939), p. 214.

[3] E. Artin, Hamburger Abhandlungen, vol. 5 (1928), p. 251; B. L. van der Waerden, *Moderne Algebra*, vol. 2; M. Deuring, *Algebren*, Ergebnisse der Mathematik, vol. 4, 1935; A. A. Albert, *Structure of Algebras*, American Mathematical Society Colloquium Publications, vol. 24, 1939.

[4] G. Köthe, Mathematische Zeitschrift, vol. 32 (1930), p. 161.

[5] Loc. cit.

[6] The treatment thus obtained seems to me simpler than Deuring's treatment.